Uniform analysis of Panel AR(1) Model by Local-to-Unity

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Abstract

This paper proposes a uniform inference method for the autoregressive coefficient of a dynamic panel model. Our method overcomes the asymptotic discontinuity of bias-corrected maximum livelihood (ML) between stationary and unit root parameter space. The proposed self-normalized t-statistic can be used to construct a confidence set that provides a uniformly valid asymptotic coverage over the whole parameter space.

Keywords: Panel AR(1), robustness, uniformity, local-to-unity, unit-root

1. Introduction

The statistical estimation and inference of autoregressive panel data model has a vast literature over the past decades. In the stable panel cases, the large sample and finite sample properties of different methods have been studied since Nickell (1981)[1]. For more persistent panel cases, Phillips and Moon (1999) [2] build up multidimensional asymptotics inference on the unit-root and near unit-root through sequential limits and joint limits. We learn from Hahn and Hahn and Kuersteiner (2002)[3] that the limiting distributions have a gap between stable and unit-root cases in terms of asymptotic bias, asymptotic variance and converging rates. This finding requires us to make stability assuption before choosing the correct confidence intervals. The similar problem has been summarized by Stock (1991)[4] in time series literature that the first-order asymptotic

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theory does not provide a suitable framework for the construction of confidence intervals because it is discontinuous at unit-root.

Although it is challenging to build a uniform inference procedure for dynamic panel model of which the parameter space include both stationarity and unit-root, we have some results in terms of time series. Stock(1991)[4] propose a local-to-unity method to construct the confidence interval of the largest autoregressive root. Followed by Andrews (1993)[5], a simulation based procedure can also overcome such discontinuity between stationary and unit-root cases under normality assumption. The uniform validity of those two methods is not proved until Mikusheva (2007). In her paper, from the view of distribution uniform approximation, the confidence set built by Stock's method and Andrews' method has correct asymptotical coverage rate uniformly over the whole parameter space that includes the stable and unit-root cases, that $\lim_{T\to\infty} \inf_{\theta_0\in\Theta} P\{\theta_0 \in C(Y)\} \geq 1-\alpha$. Mikusheva's proof is not direct to be extended into panel AR model, where the asymptotic bias emerges with time dimension expending along the cross-sectional dimension. As pointed out in Hahn and Kuersteiner (2002), the asymptotic bias can be viewed as an alternative form of the incidental parameter problem.

The challenge of extending Mikusheva's method is also mentioned in Chao and Phillips (2019) [6] and they propose an $\mathbf{M}$ statistics based on normalized Anderson-Hsiao IV procedure which corrects the asymptotic bias itself, to offer a panel uniform inference framework instead. Also, the near unity region in Mikusheva (2007) is limited to a slow shrinking speed for $0 < \alpha < 1$, while Chao and Phillips (2019) extends the validity to different shrinking speeds. However, their data-driven normalization term depends on a preliminary unit root test on the true value of θ, and the weak instrument problem results in a relative wide confidence interval by the $\mathbf{M}$ statistics as pointed out in their paper.

To address the need of uniform panel model inference and inspired by the uniform validity of

Stock's method shown by Mikuheva (2007) in time series, this paper extends the local-to-unity method to a dynamic panel data model and proposes a user-friendly one-step framework. We start with the bias-corrected MLE in the unit-root case illustrat in Hahn and Kuersteiner (2002) and propose a self-normalized t-statistics to be investigated in the stationary area and near unity area with different shrinking speeds as done in the Chao and Phillips (2019). We will prove that the proper normalization terms joins the two converging rates ($\sqrt{NT}, \sqrt{NT^2}$) and two bias-correction terms $\left(-(\hat{\theta}+1)/T, -3/(T+1)\right)$ in the same framework. Then we prove the confidence set built on this t-statistics remains correct asymptotic coverage uniformly over the whole parameter space.

The remainder of the paper proceeds as follows. Section 2 briefly introduce the model and assumptions. Section 3 proposes the self-normalized t-statistics, by which describes the discontinuity formation process from stationary to non-stationary cases. Section 4 offers the feasible self-normalized t-statistics and the construction of confidence set for the autoregressive parameter θ_T. Section 5 reports a Monte Carlo study comparing the performance of our proposed t-statistics is with the bias-correction estimator by Hahn and Kuersteiner (2002) and the **M** statistics by Chao and Philips (2019) in the stable and unit-root cases. Section 6 provides the conclusion. All the proofs and notations of the main results are attached in the Appendix.

2. Model and Assumptions

We format a linear Panel AR(1) model with cross-sectional fixed effects,

$$y_{it} = \theta_T y_{it-1} + \alpha_i + \varepsilon_{it}, \tag{1}$$

where y_{it} are univariate observables, ε_{it} are mean zero scalar error terms, θ_T is autoregressive parameter of interest, α_i is the cross-sectional fixed effect. The subscripts $i = 1, \ldots, N$ and $t = 1, \ldots T$ denote the cross-sectional unit and the time index, respectively. We denote N and T to be the dimensions of cross section and time, respectively.

We impose the following conditions that will be used in deriving the main results in the paper,

Assumption 1.

1. *ε_{it} iid $N(0, \sigma^2)$,*
2. *$\Theta_T = \{\theta_T | \theta_T = 1 - \frac{1}{q(T)}, q(T) = \frac{T^\alpha}{c}, \alpha \in [0, \infty)$ and $c \geq 0\}$*
3. *$\frac{N^\kappa}{T} \to \tau > 0$ for $\kappa \in (0, \infty)$*
4. *$y_{i,0} = 0, \forall i$*
5. *When $c = 0, \alpha_i = 0$*

The assumption 1.1 is common normality setting for constrained higher moment order and also excludes the dependency or heteroskedasticity for simplicity. Assumption 1.2 specifies the local-to-unity converging rate through α, and includes exactly (non)-stationary cases when $\alpha = 0$ or $c = 0$. Assumption 1.3 let the joint limit follow a relative speed. Since $\kappa \in (0, \infty)$, we consider all possible rates of joint convergence in terms of (N, T). Assumption 1.4 restrict the initial value to be zero. When applying our method, we could subtract the data by corresponding first period across the sections. Assumption 1.5 is a technical method to simplify our matrix expression of the model in the Appendix A, while we can show that the distribution of self-normalized t-statistics does not depend on the fixed effects.

3. Self-normalized t-statistics

In this paper, we propose the following *self-normalized t-statistics*,

$$t_{N,T}(\theta_T; \sigma^2) = \frac{\hat{\theta} - \theta_T - B_{N,T}(\theta_T, \sigma^2)}{\sqrt{V_{N,T}(\theta_T, \sigma^2)}}, \tag{2}$$

where $\hat{\theta}$ is the MLE estimator for θ_0 in (1) can be defined as

$$\hat{\theta} = \frac{\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)\left(y_{i,t} - \bar{y}_{i}\right)}{\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)^2}, \tag{3}$$

And,

$$\hat{\theta} - \theta_T := \frac{\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)\varepsilon_{it}}{\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)^2}; \tag{4}$$

$$B_{N,T}(\theta_T, \sigma^2) := \frac{E\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)\varepsilon_{it} + \frac{3}{T+1}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)^2\right)}{\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)^2} - \frac{3}{T+1}; \tag{5}$$

$$V_{N,T}(\theta_T, \sigma^2) := \frac{Var\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)\varepsilon_{it} + \frac{3}{T+1}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)^2\right)}{\left[\sum_{i=1}^{N}\sum_{t=1}^{T}\left(y_{i,t-1} - \bar{y}_{-1,i}\right)^2\right]^2}. \tag{6}$$

The asymptotic bias is the alternative form of the incidental parameter problem in panel model and its behavior is different for (non)-stationary/local-to-unity under large (N, T) asymptotics. This is the reason why it is difficult to extend the time series results found in Mikusheva (2007) to panel framework. With the adaptive bias correction term $B_{N,T}$, the proposed self-normalized t-statistics satisfies the recentering of the the moment condition in (non)-stationary cases, which is also featured by Chao and Philips' **M** statistics. The adaptive variance correction term $V_{N,T}$ joins the converging rates $(\sqrt{NT}, \sqrt{NT^2})$ of (non)-stationary cases under same framework as shown in the Appendix at Corollary 7. When properly normalized, this statistics will converge to a standard normal distribution along any path in the parameter space Θ_T.

Theorem 1. *Let $\Phi(x)$ denotes the cdf of a standard normal random variable. Under Assumptions 1, as $N, T \to \infty$, we have the uniform asymptotical distribution of the self-normalized t-statistics defined in (2),*

$$\sup_{\theta \in (-1,1]} \left| P(t_{N,T}(\theta;\sigma^2) \leq x) - \Phi(x) \right| \to 0.$$

Remark 1. *The proof of the uniform convergence here is supported by Lehmann (2004) [7, Lemma 2.6.2] with showing convergence to the same standard normal distribution at every parameter sequence in the parameter space Θ_T illustrated in the Appendix C at Lemma 3.*

Remark 2. *In the unit-root case, $c = 0, \theta_T = 1$, the bias correction term $B_{N,T}(\theta_T, \sigma^2) = -\frac{3}{T+1}$, the variance correction term $V_{N,T}(\theta_T, \sigma^2) = \frac{1}{NT^2}\frac{51}{5}(1 + o_p(1))$. Consider the representation of our self-normalized t-statistics,*

$$t_{N,T}(\theta_T;\sigma^2) = \left(\frac{\sqrt{N}T}{\sqrt{51/5}}\right)\left(\hat{\theta} - \theta_T + \frac{3}{T+1}\right)(1 + o_p(1)).$$

This shows that the bias correction and variance term in (5)–(6) are convergence rate adaptive, as it becomes equivalent to what is proposed in Hahn and Kuersteiner (2002). As mentioned previously, Chao and Philips' **M** *statistics is based on Anderson-Hsiao IV procedure, which gives well-centered moment conditions but performs poorly near(at exact) unity region because of the weak IV identification failure. Our statistics is converging to the exact limiting distribution under unit-root case, which should result in narrower confidence interval and identify the unit-root better.*

Remark 3. *In the strong local-to-unity case, $c \neq 0, \alpha > 1$, by **Corollary** 7, the variance correction term $V_{N,T}(\theta_T, \sigma^2) = \frac{1}{NT^2}\frac{51}{5}(1 + o_p(1))$. By **Corollary** 5, we have that,*

$$t_{N,T}(\theta_T;\sigma^2) = \left(\frac{\sqrt{N}T}{\sqrt{51/5}}\right)\left(\hat{\theta} - \theta_T + \frac{3}{T+1}\right)(1 + o_p(1)) + O\left(\frac{\sqrt{N}}{T^{\alpha-1}}\right).$$

where the additional remainder term comes from

$$\frac{B_{N,T}(\theta_T, \sigma^2)}{\sqrt{V_{N,T}(\theta_T, \sigma^2)}} = \underbrace{\frac{-3/(T+1)}{\sqrt{V_{N,T}(\theta_T, \sigma^2)}}}_{->\ \textit{Unit root case}} + O\left(\frac{\sqrt{N}}{T^{\alpha-1}}\right).$$

By Assumption 1 $\frac{N^\kappa}{T} \to \tau > 0$ and thus $O\left(\frac{\sqrt{N}}{T^{\alpha-1}}\right) = O\left(\tau^{\alpha-1}N^{\frac{1}{2}-\kappa(\alpha-1)}\right)$. We conclude (i) if $\kappa = \frac{1}{2(\alpha-1)}$, $O\left(\frac{\sqrt{N}}{T^{\alpha-1}}\right) = O(1)$; (ii) if $\kappa > \frac{1}{2(\alpha-1)}$, $O\left(\frac{\sqrt{N}}{T^{\alpha-1}}\right) = o(1)$; (iii) if $\kappa < \frac{1}{2(\alpha-1)}$, $O\left(\frac{\sqrt{N}}{T^{\alpha-1}}\right) = O\left(N^{\frac{1}{2}-\kappa(\alpha-1)}\right)$. This shows to our local-to-unity statistics can be equivalent to the unit-root situation, if we make sure the relation between local-to-unity speed ($\alpha > 1$) and dimension growth rate (κ) guarantee the case(ii).

Remark 4. *In the semi-strong local-to-unity case, $c \neq 0, \alpha = 1$, by **Corollary** 7, the variance correction term $V_{N,T} = \frac{\frac{1}{NT^4}2\sigma^4 B_c T^2}{\sigma^4 K_c^2 + o_p(1)} = \frac{1}{NT^2}\frac{2B_c}{K_c^2} + o_p(1)$, where A_c, B_c, K_c are the constant depending on c defined in the Corollary 5. Also we have that,*

$$t_{N,T}(\theta_T;\sigma^2) = \left(\frac{\sqrt{N}T}{\sqrt{2B_c/K_c^2}}\right)\left(\hat{\theta} - \theta_T + \frac{3}{T+1}\right)(1 + o_p(1)) + O\left(\sqrt{N}\right).$$

where the additional remainder term comes from

$$\frac{B_{N,T}(\theta_T,\sigma^2)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} = \frac{-3/(T+1)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} + \frac{\sqrt{N}A_c}{\sqrt{2B_c}} + o(\sqrt{N}).$$

This shows our semi-strong local-to-unity case cannot be approximated by unit-root situation, for the local-to-unity speed is not fast enough to converge to the unity.

Remark 5. *In the weak local-to-unity case, $c \neq 0, 0 < \alpha < 1$, by **Corollary** 7, the variance correction term $V_{N,T} = \frac{2c}{NT^{1+\alpha}} + o_p(1)$. Also we have that,*

$$t_{N,T}(\theta_T;\sigma^2) = \left(\frac{\sqrt{NT^{1+\alpha}}}{\sqrt{2c}}\right)\left(\hat{\theta} - \theta_T + \frac{3}{T+1}\right)(1+o_p(1)) + O\left(\sqrt{NT^{\alpha-1}}\right).$$

where the additional remainder term comes from

$$\frac{B_{N,T}(\theta_T,\sigma^2)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} = \frac{-3/(T+1)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} + \frac{\sqrt{NT^{\alpha-1}}}{\sqrt{2c}} + o(\sqrt{NT^{\alpha-1}})$$

Remark 6. *In the stationary case, $\alpha = 0$, $|\theta_T| = |1-c| < 1$. By **Corollary** 7, the variance correction term $V_{N,T} = \frac{1-(1-c)^2}{NT} + o_p(1)$. Also we have that,*

$$t_{N,T}(\theta_T;\sigma^2) = \sqrt{\frac{NT}{1-(1-c)^2}}\left(\hat{\theta} - \theta_T + \frac{3}{T+1}\right)(1+o_p(1)) + O\left(\sqrt{NT^{-1}}\right)$$

where the additional remainder term comes from

$$\frac{B_{N,T}(\theta_T,\sigma^2)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} = \frac{-3/(T+1)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} + \frac{-1/c + 3/(1-(1-c)^2)}{\sqrt{T/(1-(1-c)^2)}}\sqrt{N} + o\left(\sqrt{NT^{-1}}\right)$$

We expand the remainder term and plugin the identity $1 - c = \theta_T$,

$$t_{N,T}(\theta_T;\sigma^2) = \underbrace{\sqrt{\frac{NT}{1-\theta_T^2}}\left(\hat{\theta} + \frac{\theta_T}{T} - \theta_T + \frac{1}{T}\right)}_{\textit{stationary case}} + \sqrt{\frac{NT}{1-\theta_T^2}}\left(\hat{\theta} - \theta_T + \frac{3}{T+1}\right)o_p(1) + \sqrt{\frac{N}{T}}o(1)$$

We can learn from the stationary case, our self-normalized t-statistics converges to Hahn and Kuersteiner (2002)'s if N/T grows to a constant as assumed in their paper.

4. Confidence Set with Asymptotic Uniform Validity

Using the uniform convergence in Theorem 1, one can construct the following level $1-a$ confidence set by inverting test (Lehmann (1997, p. 90)[8]) as in Mikusheva (2007)[9], where $z_{1-a/2}$ is the standard normal $1-a/2$ quantile:

$$C_{N,T} := \left\{\theta \in (-1,1] : \left|t_{N,T}(\theta;\sigma^2)\right| < z_{1-a/2}\right\}. \tag{7}$$

It is immediate from Theorem 1 that,

Corollary 1. *Under Assumption 1 and consider the confidence set in (7),*

$$\lim_{N,T\to\infty} \inf_{\theta\in(-1,1]} P_{\theta,\sigma^2}\left(\theta \in C_{N,T}\right) \geq 1-a, \tag{8}$$

We say $C_{N,T}$ is an asymptotic confidence set at a confidence level $1-a$, or to have a uniform asymptotic coverage probability $1-a$.

4.1. σ^2 estimation and feasible inference

The proposed self-normalized t-statistics in the previous section is infeasible in terms of the bias and variance normalization parts. Now consider the MLE variance estimation,

$$\hat{\alpha}_i = \frac{1}{T}\sum_t y_{it} - \hat{\theta} y_{it-1},$$

$$\hat{\sigma}^2 = \frac{1}{NT}\sum_{it}[y_{it} - (\hat{\theta} y_{it-1} + \hat{\alpha}_i)]^2.$$

Although the MLE $\hat{\theta}$ is downward biased but consistent, the resulting $\hat{\sigma}^2$ is showing uniformly consistent in the following theorem.

Theorem 2. *Under Assumption 1,*

$$\lim_{N,T\to\infty} \sup_{\theta_T\in\Theta_T} P\left\{\left|\frac{\hat{\sigma}^2}{\sigma^2} - 1\right| > \epsilon\right\} = 0, \forall\epsilon > 0. \tag{9}$$

Proof in Appendix D.

Our findings in Theorems 1 and 2 lead to the following uniform convergence result to construct a feasible self-normalized t-statistics and feasible construction of confidence set.

Theorem 3. *Under the Assumption 1 and using the consistent estimator $\hat{\sigma}^2$ in (9), we denote*

$$\hat{B}_{N,T}(\theta) = \frac{N\hat{\sigma}^2 tr[G_T(\theta)]}{\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1}-\bar{y}_{-1,i})^2} - \frac{3}{T+1},$$

$$\hat{V}_{N,T}(\theta) = \frac{N\hat{\sigma}^4 tr[M_T^2(\theta)]}{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{it-1}-\bar{y}_{-1,i})^2\right)^2}.$$

Then as $N, T \to \infty$, the asymptotic distribution of a feasible self-normalized t-statistic satisfies:

$$\hat{t}_{N,T}(\theta_T) = \frac{\hat{\theta} - \theta_T - \hat{B}_{N,T}(\theta_T)}{\sqrt{\hat{V}_{N,T}(\theta_T)}} \tag{10}$$

$$\sup_{\theta\in(-1,1]} \left|P(\hat{t}_{N,T}(\theta) \le x) - \Phi(x)\right| \to 0.$$

Proof in Appendix E.

Using the result in Theorem 3, we can construct the following confidence set, where $z_{1-a/2}$ is the standard normal $1-a/2$ quantile:

$$\hat{C}_{N,T} := \left\{\theta \in (-1,1] : \left|\hat{t}_{N,T}(\theta)\right| < z_{1-a/2}\right\}, \tag{11}$$

It is immediate from Theorem 3 that,

Corollary 2. *Under Assumption 1 and consider the confidence set in (11),*

$$\lim_{N,T\to\infty} \inf_{\theta\in(-1,1]} P_{\theta,\sigma^2}\left(\theta \in \hat{C}_{N,T}\right) \ge 1-a, \tag{12}$$

We say $\hat{C}_{N,T}$ is an asymptotic confidence set at a confidence level $1-a$, or to have a uniform asymptotic coverage probability $1-a$.

5. Simulation

We set $\sigma = 2$ and choose different N, T, θ_T to simulate 1000, 2000, 5000 samples generated from our model (1) to show the feasible confidence set $\hat{C}_{N,T}$ in (11) has correct uniform asymptotical coverage rate. The performance of our proposed t-statistics is also compared with the bias-correction estimator by Hahn and Kuersteiner (2002)[3] and **M** statistics by Chao and Philips (2019) [6] in both stable case and unit-root case denoted as *HKstable*, *HKnonstable*, *Mstat*. The algorithm of constructing $\hat{C}_{N,T}$ in (11) is as follows:

Algorithm 1: Simulation

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1 Make a fine grid of parameter space, e.g. $\Theta = [0.90, 1]$. Let's say the grid as Θ_g where g is the number of grids;

2 Pick up the one element of Θ_g, let's say it is $\theta_{(i)}$;

3 Use $\theta_{(i)}$ to generate data and construct MLE $\hat{\theta}, \hat{\sigma}^2$;

4 Pick one element of Θ_g, let's say it is $\theta_{(j)}$. Use $\theta_{(j)}$ to compute the matrix D_T, G_T, M_T and,

$$\hat{t}_{N,T}(\theta_{(j)}) = \frac{\hat{\theta} - \theta_{(j)} - \hat{B}_{N,T}(\theta_{(j)})}{\sqrt{\hat{V}_{N,T}(\theta_{(j)})}};$$

$$\hat{B}_{N,T}(\theta_{(j)}) := \frac{N\hat{\sigma}^2 tr(G_T)}{\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})^2} - \frac{3}{T+1};$$

$$\hat{V}_{N,T}(\theta_{(j)}) := \frac{N\hat{\sigma}^4 2tr(M_T^2)}{\left(\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})^2\right)^2};$$

5 Keep $\theta_{(j)}$ in $C_{N,T}$ if $\left|t_{N,T}(\theta_{(j)};\hat{\sigma}^2)\right| < z_{1-\alpha/2}$;

6 Repeat 4 – 5, with the next element $\theta_{(j)}$ until the last left grid and finish the construction of $C_{N,T}$.;

7 Repeat 3 – 6 and count the percentage rate that $\theta_{(i)} \in C_{N,T}$.

In the Figure 1-3, the lines are finite sample coverage rate of three confidence sets for different true autoregressive parameter $\theta_T = \theta$. The solid blue line represents our proposed confidence set $\hat{C}_{N,T}$ in (11). The yellow dotted line represents descripts the confidence set constructed by *HKstable* $\left[\hat{\theta} + \frac{1}{T}(1+\hat{\theta}) \pm \frac{\sqrt{(1-\hat{\theta}^2)}}{\sqrt{NT}} \times z_{1-\alpha/2}\right]$; the green dashed line is the confidence set constructed by *HKnonstable* $\left[\hat{\theta} + \frac{3}{T+1} \pm \frac{1}{\sqrt{NT^2}}\sqrt{\frac{51}{5}} \times z_{1-\alpha/2}\right]$; the red dashdot line presents the confidence set constructed by *Mstat* $\left\{\theta \in (-1,1] : -z_{1-\alpha/2} \leq \mathbf{M}(\theta) \leq z_{1-\alpha/2}\right\}$; the black horizontal line is the 95%

nominal coverage rate.

We can learn from the figures that the *HKstable* works worst when the true autoregressive parameter is closer to unity, especially when dimension ratio N/T is also large its coverage rate is far below 50%. Within the range of close to 1, the *HKnonstable* shows better coverage as the true autoregressive parameter moving toward unit-root. Although it is more accurate in finite sample to use *HKnonstable* to construct the confidence interval, the method crucially relies on prior information about the non-stationarity. It illustrates the discontinuity between stable and nonstable situations. The Chao and Philips' **M** statistics shows a strong coverage rate over the whole parameter range. However, we can find that the strength tends to exceed the nominal level because of wide confidence interval at the cost of weak identification. In contrast, out proposed confidence interval using the self-normalized t-statistic does not require the knowledge of the degree of persistency and shows a reliable performance in terms of the finite sample coverage rate over the whole parameter space $[0.9, 1]$.

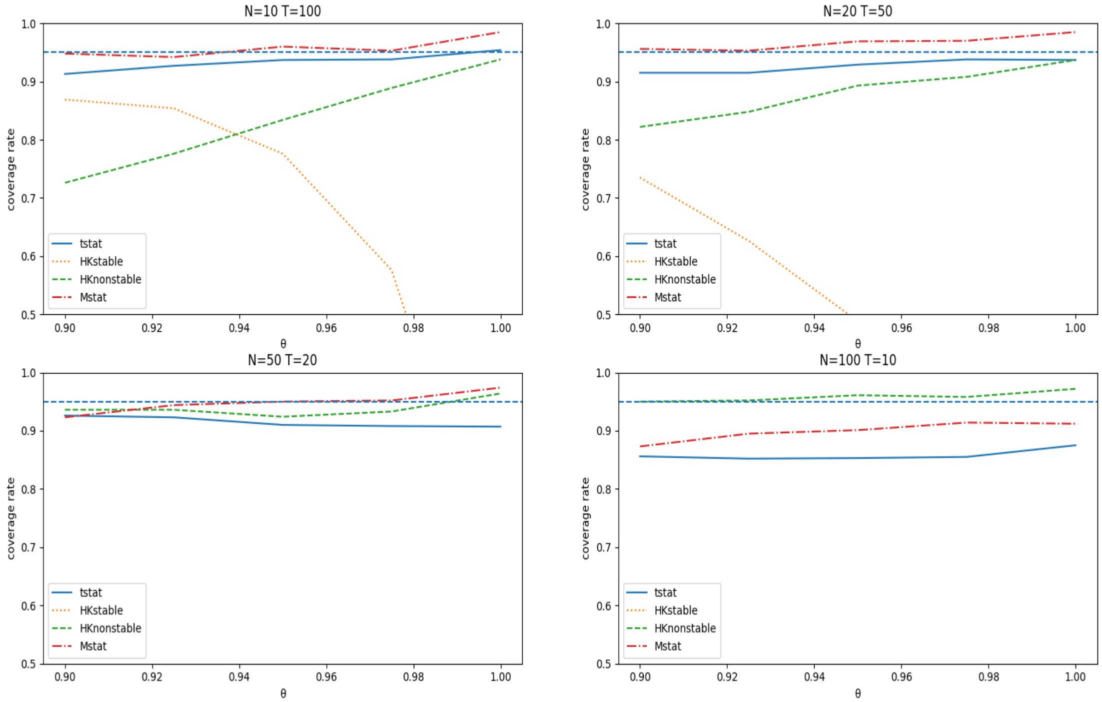

Figure 1: Simulation size 1000

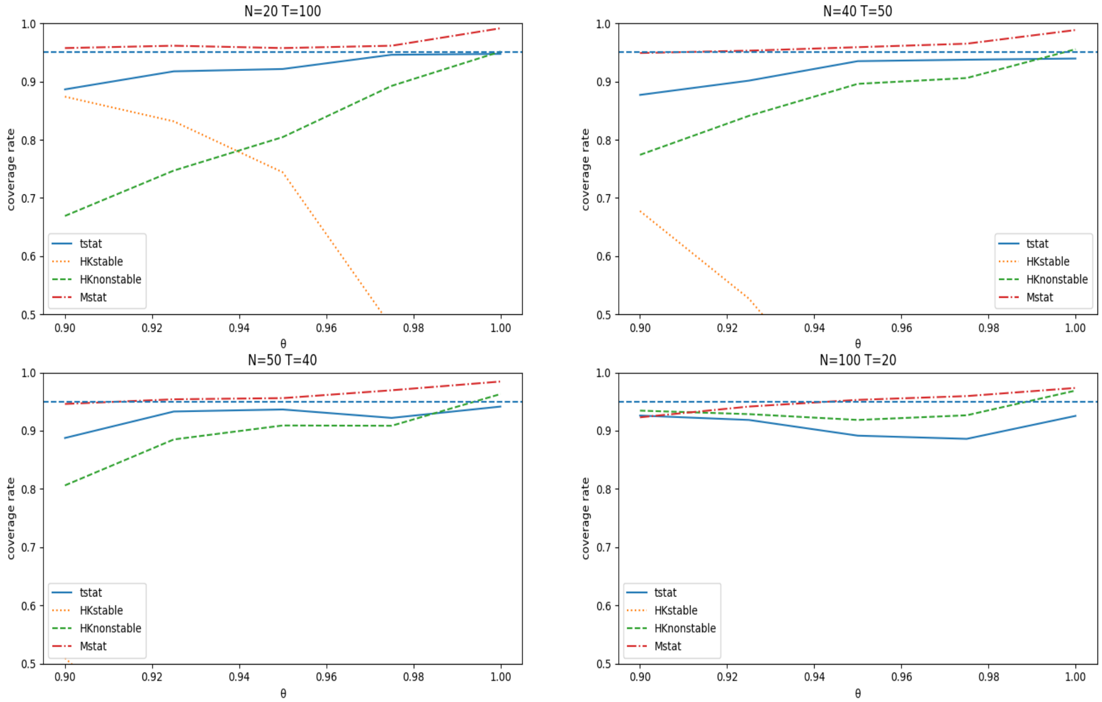

Figure 2: Simulation size 2000

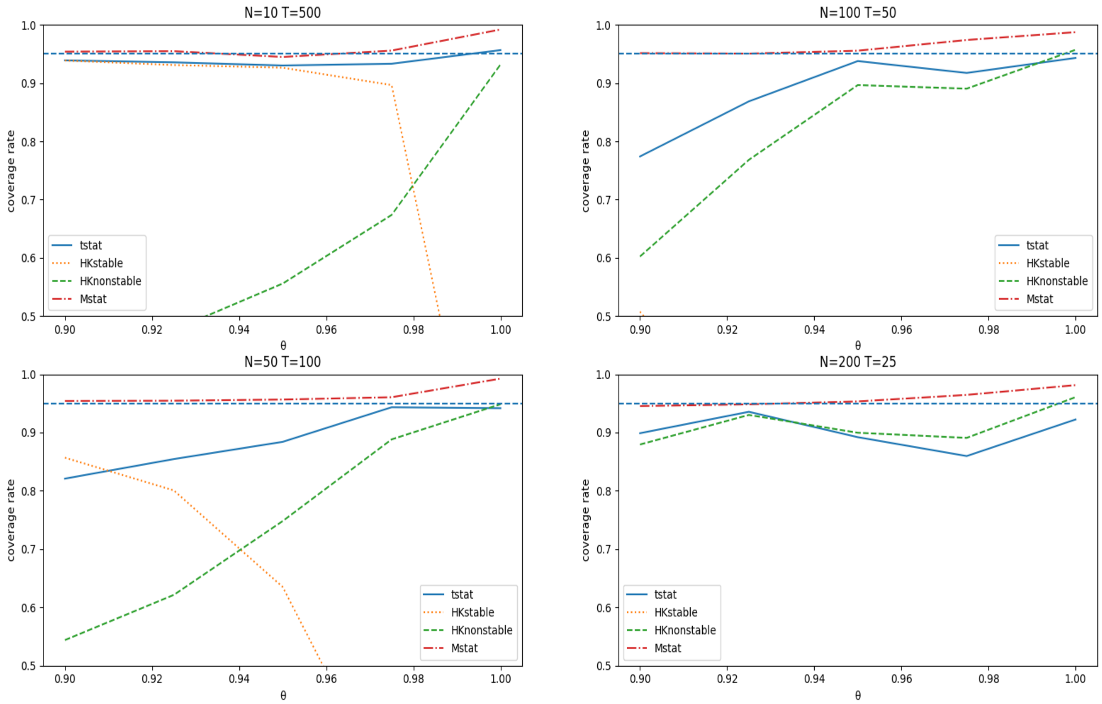

Figure 3: Simulation size 5000

6. Empirical Study

We conduct a unit-root test using proposed self-normalized t statistics for the purchasing power parity (PPP) hypothesis on the US real exchange rates data. When the null of a unit root is not rejected for the log real exchange rates, it is inferred that the PPP hypothesis does not hold. This application of testing unit-root null for the PPP hypothesis has been adopted among the literatures for long time. And most previous studies could not find evidence in favor of the PPP hypothesis, while some of them report evidence supporting the PPP hypothesis as mentioned in Choi (2001) [10]. The data we are to investigate is same as in Choi(2001) for comparison, which are the monthly US real exchange rates vs. the Canadian dollar, German Mark, Japanese Yen, French Franc, British Pound and the Swiss Franc. The period ranges from 1973:3 to 1996:3. We can learn from the Table

1, the 95% confidence interval constructed by our proposed self-normalized t-statistics does not contain the unit-root. It means we can not accept the null that the PPP hypothesis does not hold.

Result		
tstat	HKstable	HKnonstable
(0.632, 0.664)	(0.605, 0.678)	(0.639, 0.656)

Table 1: 95% Confidence Interval of AR(1) coefficient

7. Conclusion

In this paper, we take the local-to-unity method to study the near unit-root parameter space of Panel AR(1) model. A self-normalized t-statistics is introduced to bridge the discontinuity between the $\sqrt{NT}$ stable converging rate and $\sqrt{NT^2}$ nonstable converging rate as well as the asymptotic bias. The simulation study confirms the validity of our uniformly correct confidence set compared with the Hahn and Kuersteiner (2002)'s exact inference and Chao and Philips (2019)'s procedure. In practice, our method provide the appliers with a parameter insensitive confidence set construction one-step framework to obtain the valid inference for Panel AR(1) model with unknown persistence. Future research could study the weaker assumption and compare the efficiency of confidence set with other procedures in the recent literature when near unity.

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Appendix A. Matrix Expressions

When $c \neq 0$, model(1) can be represented as a shifted latent process,

$$\begin{aligned} y_{i,t} &= y_{i,t}^* + \frac{\alpha_i}{1-\theta_T}, \\ y_{i,t}^* &= \theta_T y_{i,t-1}^* + \varepsilon_{it}. \end{aligned} \tag{A.1}$$

We can easily show that the distribution of self-normalized t-statistics does not depend on the fixed effects α_i because,

$$\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{-1,i})^2 = \sum_{t=1}^{T} \left(y_{i,t-1}^* - \bar{y}_{-1,i}^*\right)^2$$
$$\sum_{t=1}^{T} (y_{i,t-1} - \bar{y}_{-1,i})(y_{i,t} - \bar{y}_i)) = \sum_{t=1}^{T} (y_{i,t-1}^* - \bar{y}_{-1,i}^*)(y_{i,t}^* - \bar{y}_i^*))$$

When $c = 0$, we have to assume $\alpha_i = 0$ or design the model to be $y_{i,t} = \theta_T y_{i,t-1} + (1-\theta_T)\alpha_i + \varepsilon_{it}$ as in Cao,Kao,Hwang (2020) to ensure the self-normalized t-statistics does not depend on the fixed effects.

Start from the proof here, we can simply take $\alpha_i = 0$ and denote the $y_{i-} = (y_{i,0}, \cdots, y_{i,T-1})'$, $y_i = (y_{i,0}, \cdots, y_{i,T})'$, $\varepsilon_i = (\varepsilon_{i,1}, \cdots, \varepsilon_{i,T})'$, $H_T = I_T - \frac{1}{T} 1_T 1_T'$. Then (1) can be represented in the matrix form,

$$y_{i-} = \begin{pmatrix} 1 \\ \theta_T \\ \theta_T^2 \\ \vdots \\ \theta_T^{T-1} \end{pmatrix} y_{i,0} + \begin{pmatrix} 0 \\ 1 \\ 1+\theta_T \\ \vdots \\ 1+\theta_T+\theta_T^{T-2} \end{pmatrix} \alpha_i + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \theta_T & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots \\ \theta_T^{T-2} & \theta_T^{T-3} & \dots & 0 \end{pmatrix} \varepsilon_i$$

$$:= A_0 y_{i,0} + A_\alpha \alpha_i + A_T \varepsilon_i.$$

$$\sum_{t=1}^{T}(y_{i,t-1}-\bar{y}_{-1,i})^2 = y_{i-}'H_T y_{i-} = \varepsilon_i' D_T' D_T \varepsilon$$
$$\sum_{t=1}^{T}(y_{i,t-1}-\bar{y}_{-1,i})(\varepsilon_{i,t}-\bar{\varepsilon}_i) = \varepsilon_i' D_T \varepsilon_i$$

where D_T is the short symbol of $D_T(\theta_T) = H_T A_T$,

$$D_T(\theta_T) = \begin{bmatrix} 0-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & 0-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & \cdots & 0 \\ 1-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & 0-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & \cdots & 0 \\ \theta_T-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & 1-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots \\ \theta_T^{T-2}-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & \theta_T^{T-3}-\frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & ... & 0 \end{bmatrix}. \tag{A.2}$$

$M_T(\theta) = (G_T(\theta) + G_T'(\theta))/2$, $G_T(\theta) = D_T(\theta) + \frac{3}{T+1}D_T'(\theta)D_T(\theta)$,

Appendix B. Technical Lemmas

We first provide some necessary technical lemmas from Han and Kuersteinder (2002)[3] and Chao and Phillips (2020)[11].

Lemma 1 (Taylor Expansion). *Let x be some positive value converging to zero,*

1. $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^i}{i!} + O(x^{i+1})$
2. $(1-e^{-x})^{-1} = x^{-1}(1 + \frac{1}{2}x + \frac{1}{12}x^2 + 0 * x^3 + \cdots + O(x^{n+1})) := x^{-1}E_1(x,n)$
3. $(1-e^{-x})^{-2} = x^{-2}(1 + x + \frac{5}{12}x^2 + \frac{1}{12}x^3 + \cdots + O(x^{n+1})) := x^{-2}E_2(x,n)$
4. $(1-e^{-x})^{-3} = x^{-3}(1 + \frac{3}{2}x + x^2 + \frac{3}{8}x^3 + \cdots + O(x^{n+1})) := x^{-3}E_3(x,n)$
5. $(1-e^{-x})^{-4} = x^{-4}(1 + 2x + \frac{11}{6}x^2 + x^3 + \cdots + O(x^{n+1})) := x^{-4}E_4(x,n)$
6. $(1-e^{-x})^{-5} = x^{-5}(1 + \frac{5}{2}x + \frac{35}{12}x^2 + \frac{25}{12}x^3 + \cdots + O(x^{n+1})) := x^{-5}E_5(x,n)$

And for $T^m \sum_{n=m}^{\infty} \left(\frac{cT}{T^\alpha}\right)^{n-m} \left(\frac{k(-x)^{n+t}}{(n+t)!}\right)$*, with* $m \geq 0, 0 < \alpha < 1$*, we have*

$$\begin{aligned} & T^m \sum_{n=m}^{\infty} \left(\frac{cT}{T^\alpha}\right)^{n-m} \left(\frac{k(-x)^{n+t}}{(n+t)!}\right) \\ &= \frac{T^{(t+m)\alpha-t}}{c^{t+m}} \left[ke^{-xc/T^{\alpha-1}} - \sum_{n=0}^{m+t-1} \left(\frac{c}{T^{\alpha-1}}\right)^n \left(\frac{k(-x)^n}{n!}\right)\right] \\ &= ke^{-xc/T^{\alpha-1}} \frac{T^{(t+m)\alpha-t}}{c^{t+m}} - \sum_{n=0}^{m+t-1} \frac{k(-x)^n}{n!} \frac{T^{n(1-\alpha)+(t+m)\alpha-t}}{c^{t+m-n}} \\ &= -\frac{k(-x)^{m+t-1}}{(m+t-1)!} \frac{T^{m-1+\alpha}}{c} - \frac{k(-x)^{m+t-2}}{(m+t-2)!} \frac{T^{m-2+2\alpha}}{c^2} + o(T^{m-2+2\alpha}) \end{aligned} \tag{B.1}$$

Lemma 2 (Lemma 10 in Hahn and Kuersteiner (2002)[3]).

(i) When $c \neq 0, \alpha \in (0, \infty)$*,*

$$tr(D_T) = T \sum_{n=1}^{\infty} c^{n-1} \left(\frac{T}{T^\alpha}\right)^{n-1} \left(-\frac{(-1)^{n+1}}{n+1!}\right) + O(1);$$

$$tr(D_T D_T^{'}) = T^2 \sum_{n=2}^{\infty} c^{n-2} \left(\frac{T}{T^\alpha}\right)^{n-2} \left(\frac{(-2)^{n+1}/2}{n+1!} + \frac{(-2)^n/4}{n!} - \frac{2(-1)^{n+1}}{n+1!}\right) + O(T);$$

$$tr(D_T^2) = T^2 \sum_{n=2}^{\infty} c^{n-2} \left(\frac{T}{T^\alpha}\right)^{n-2} \left(\frac{(-2)(-1)^n}{n!} + \frac{(-2)^{n+2} - 2(-1)^{n+2}}{n+2!} - \frac{2(-1)^{n+1}}{n+1!}\right) + o(T^2);$$

$$\begin{aligned} tr[(D_T^{'} D_T)^2] = T^4 \sum_{n=4}^{\infty} c^{n-4} \left(\frac{T}{T^\alpha}\right)^{n-4} & \left[\frac{(-2)^{n-1}/2}{n-1!} + \frac{(-2)^n/4 + (-4)^n/16 - 2(-1)^n + (-2)^n}{n!}\right. \\ & + \frac{-3(-1)^{n+1} + 2(-2)^{n+1} + (-4)^{n+1}/4 - (-3)^{n+1}}{(n+1)!} \\ & \left.+ \frac{-(-2)^{n+2}/2 + (-4)^{n+2}/4 - 6(-1)^{n+2} + 6(-2)^{n+2} - 2(-3)^{n+2}}{(n+2)!}\right] + o(T^4); \end{aligned}$$

$$\begin{aligned} tr(D_T^{'} D_T^2) = {} & T^3 * 0 + \sum_{n=4}^{\infty} \frac{T^n}{q(T)^{n-3}} * \left[\frac{(-2)^{n-1}/4}{(n-1)!} + \frac{0.75(-2)^n - 1.5(-1)^n}{n!}\right. \\ & \left. - \frac{(-3)^{n+1}/4 + 11(-1)^{n+1}/4 - 5(-2)^{n+1}/4}{(n+1)!} + \frac{2.5(-2)^{n+2} - 3.5(-1)^{n+2} - (-3)^{n+2}/2}{(n+2)!}\right] \\ & - \frac{1}{12}T^2 + \sum_{n=3}^{\infty} \frac{T^n}{q(T)^{n-2}} * \left[\frac{3(-2)^{n-1}/4}{(n-1)!} + \frac{-3.5(-1)^n + 2(-2)^n}{n!}\right. \\ & \left. - \frac{5.25(-1)^{n+1} - 3 * (-2)^{n+1} + 0.75(-3)^{n+1}}{(n+1)!} + \frac{6.5(-2)^{n+2} - 8.5(-1)^{n+2} - (-3)^{n+2}1.5}{(n+2)!}\right] + o(T^2); \end{aligned}$$

(ii) When $c \neq 0, \alpha = 0$,

$$tr(D_T) = -\frac{1}{c} + o(1);$$
$$tr(D_T D_T^{'}) = \frac{T}{1-(1-c)^2} + O(1);$$
$$tr(D_T^2) = -\frac{1}{c^2} + o(1);$$
$$tr[(D_T^{'} D_T)^2] = \frac{2T}{(1-(1-c)^2)^3} - \frac{T}{(1-(1-c)^2)^2} + o(T);$$
$$tr(D_T^{'} D_T^2) = \frac{(1-c)T}{(1-(1-c)^2)^2} + o(T);$$

Proof 1. *Denote $M_k = \sum_{j=0}^{k} \theta_T^j$ only in this proof session, and decompose $D_T = D_1 - D_2$ as below,*

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \theta_T & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \theta_T^{T-2} & \theta_T^{T-3} & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & \cdots & \frac{1}{T} & 0 \\ \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & \cdots & \frac{1}{T} & 0 \\ \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & \cdots & \frac{1}{T} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-2}) & \frac{1}{T}(1+\theta_T+\cdots+\theta_T^{T-3}) & \cdots & \frac{1}{T} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \theta_T & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \theta_T^{T-2} & \theta_T^{T-3} & \cdots & 1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{T}M_{T-2} & \frac{1}{T}M_{T-3} & \cdots & \frac{M_0}{T} & 0 \\ \frac{1}{T}M_{T-2} & \frac{1}{T}M_{T-3} & \cdots & \frac{M_0}{T} & 0 \\ \frac{1}{T}M_{T-2} & \frac{1}{T}M_{T-3} & \cdots & \frac{M_0}{T} & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{T}M_{T-2} & \frac{1}{T}M_{T-3} & \cdots & \frac{M_0}{T} & 0 \end{bmatrix} := D_1 - D_2$$

and can easily show,

$$D_1' D_2 = D_2' D_1 = D_2' D_2 \tag{B.2}$$

Also we convert the $\theta_T = 1 - \frac{1}{q(T)}$ as $e^{-1/q(T)}$.

(0) When $c = 0$, the result directly follows by Lemma 10,11,12 from Hahn and Kuersteinder (2002)[3].

(1) Now we consider the case when $c \neq 0$.

For $\alpha \in (0, \infty)$,

$$\begin{aligned} tr(D_T) &= -\frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{t-1}\theta_T^{t-1-j} = -\frac{1}{T}\sum_{t=1}^{T}\frac{1-\theta_T^{t-1}}{1-\theta_T} = -\frac{1}{T}\frac{T - \frac{1-\theta_T^T}{1-\theta_T}}{1-\theta_T} \\ &= -\frac{1}{1-\theta_T} + \frac{1-\theta_T^T}{T(1-\theta_T)^2} \\ &= -q(T)(1+O(\frac{1}{q(T)})) + \frac{q(T)^2}{T}(1+\frac{1}{q(T)} + O(\frac{1}{q(T)^2}))\sum_{n=1}^{\infty} -\frac{(-T)^n}{n!q(T)^n} \\ &= q(T)*0 + \sum_{n=1}^{\infty}\frac{T^n}{q(T)^{n-1}}\left(-\frac{(-1)^{n+1}}{n+1!}\right) + O(1) \end{aligned}$$

When $\alpha = 0$, $\theta_T = 1 - c$,

$$\begin{aligned} tr(D_T) &= -\frac{1}{1-\theta_T} + \frac{1-\theta_T^T}{T(1-\theta_T)^2} \\ &= -\frac{1}{c} + \frac{1-(1-c)^T}{Tc^2} \\ &= -\frac{1}{c} + O(T^{-1}) \end{aligned}$$

Denote $S_k = \sum_{j=0}^{k} (\theta_T^j)^2$, *when* $\alpha \in (0, \infty)$,

$$\begin{aligned} tr(D_T D_T^{'}) &= \sum_{ij} d_{ij}^2 = [\sum_{k=0}^{T-2} S_k] - \frac{2}{T}[\sum_{k=0}^{T-2} M_k^2] + \frac{T}{T^2}[\sum_{k=0}^{T-2} M_k^2] \\ &= [\sum_{k=0}^{T-2} \frac{1-\theta_T^{2(k+1)}}{1-\theta_T^2}] - \frac{1}{T}[\sum_{k=0}^{T-2} \frac{1-2\theta_T^{k+1}+\theta_T^{2(k+1)}}{(1-\theta_T)^2}] \\ &= \frac{T-1-\frac{\theta_T^2-\theta_T^{2T}}{1-\theta_T^2}}{1-\theta_T^2} - \frac{T-1-2\frac{\theta_T-\theta_T^T}{1-\theta_T}+\frac{\theta_T^2-\theta_T^{2T}}{1-\theta_T^2}}{T(1-\theta_T)^2} \\ &= \left\{ \frac{T-1}{1-\theta_T^2} - \frac{\theta_T^2}{(1-\theta_T^2)^2} - \frac{T-1}{T(1-\theta_T)^2} + \frac{2\theta_T}{T(1-\theta_T)^3} - \frac{\theta_T^2}{T(1-\theta_T)^2(1-\theta_T^2)} \right\} \\ &+ \frac{\theta_T^{2T}}{T(1-\theta_T)^2(1-\theta_T^2)} + \frac{\theta_T^{2T}}{(1-\theta_T^2)^2} - \frac{2\theta_T^T}{T(1-\theta_T)^3} \\ &= \left(\frac{q(T)^3+q(T)^2+q(T)}{T} + q(T)^2 + q(T) + Tq(T) + T \right) * 0 + \\ &\sum_{n=2}^{\infty} \frac{T^n}{q(T)^{n-2}} \left(\frac{-(-2)^n}{n+1!} + \frac{(-2)^{n-2}}{n!} - \frac{2(-1)^{n+1}}{n+1!} \right) + O(T) \end{aligned}$$

where $O(T) = \sum_{n=1}^{\infty} \frac{T^n}{q(T)^{n-1}}(\cdots) + \sum_{n=0}^{\infty} \frac{T^n}{q(T)^n}(\cdots) + o(1)$.

When $\alpha = 0$,

$$\begin{aligned} tr(D_T D_T^{'}) &= \left\{ \frac{T-1}{1-\theta_T^2} - \frac{\theta_T^2}{(1-\theta_T^2)^2} - \frac{T-1}{T(1-\theta_T)^2} + \frac{2\theta_T}{T(1-\theta_T)^3} - \frac{\theta_T^2}{T(1-\theta_T)^2(1-\theta_T^2)} \right\} \\ &+ \frac{\theta_T^{2T}}{T(1-\theta_T)^2(1-\theta_T^2)} + \frac{\theta_T^{2T}}{(1-\theta_T^2)^2} - \frac{2\theta_T^T}{T(1-\theta_T)^3} \\ &= \left\{ \frac{T-1}{1-(1-c)^2} - \frac{(1-c)^2}{(1-(1-c)^2)^2} - \frac{T-1}{Tc^2} + \frac{2(1-c)}{Tc^3} - \frac{(1-c)^2}{Tc^2(1-(1-c)^2)} \right\} \\ &+ \frac{(1-c)^{2T}}{Tc^2(1-(1-c)^2)} + \frac{(1-c)^{2T}}{(1-(1-c)^2)^2} - \frac{2(1-c)^T}{Tc^3} \\ &= \left\{ \frac{T-1}{1-(1-c)^2} - \frac{(1-c)^2}{(1-(1-c)^2)^2} - \frac{1}{c^2} + O(T^{-1}) \right\} + o(T^{-1}) + o(1) - o(T^{-1}) \end{aligned}$$

When $\alpha \in (0, \infty)$,

$$\begin{aligned}
trace(D_T^2) &= tr(D_1D_1) - 2tr(D_1D_2) + tr(D_2D_2) \\
&= [0] - [\frac{2}{T}\sum_{k=0}^{T-3} M_k M_{T-3-k}] + [\frac{1}{T^2}(\sum_{k=0}^{T-2} M_k)^2] \\
&= -\frac{2}{T}\sum_{k=0}^{T-3} \frac{1-\theta_T^{k+1}}{1-\theta_T}\frac{1-\theta_T^{T-2-k}}{1-\theta_T} + \frac{1}{T^2}\left(\sum_{k=0}^{T-2}\frac{1-\theta_T^{k+1}}{1-\theta_T}\right)^2 \\
&= -2\frac{T - T\theta_T - 2 + T\theta_T^{T-1} - T\theta_T^T + 2\theta_T^T}{T(1-\theta_T)^3} + \\
&\frac{T^2 - 2T + 1 + T^2\theta_T^2 + 2T\theta_T - 2T^2\theta_T + \theta_T^{2T} - 2\theta_T^T + 2T\theta_T^T - 2T\theta_T^{T+1}}{T^2(1-\theta_T)^4} \\
&= \left(q(T)^4 + q(T)^3 + q(T)^2 + q(T) + \frac{q(T)^4 + q(T)^3 + q(T)^2 + q(T)}{T} + \frac{q(T)^4}{T^2}\right) * 0 \\
&+ \sum_{n=2}^{\infty} \frac{T^n}{q(T)^{n-2}}\left(\frac{(-2)(-1)^n}{n!} + \frac{(-2)^{n+2} - 2(-1)^{n+2}}{n+2!} - \frac{2(-1)^{n+1}}{n+1!}\right) + o(T^2)
\end{aligned}$$

where $o(T^2) = \sum_{n=1}^{\infty} \frac{T^n}{q(T)^{n-1}}(\cdots) + \sum_{n=0}^{\infty} \frac{T^n}{q(T)^n}(\cdots) + o(1)$.
when $\alpha = 0$,

$$\begin{aligned}
trace(D_T^2) &= -2\frac{T - T\theta_T - 2 + T\theta_T^{T-1} - T\theta_T^T + 2\theta_T^T}{T(1-\theta_T)^3} + \\
&\frac{T^2 - 2T + 1 + T^2\theta_T^2 + 2T\theta_T - 2T^2\theta_T + \theta_T^{2T} - 2\theta_T^T + 2T\theta_T^T - 2T\theta_T^{T+1}}{T^2(1-\theta_T)^4} \\
&= -2\frac{Tc - 2 + T(1-c)^{T-1} - T(1-c)^T + 2(1-c)^T}{Tc^3} + \\
&\frac{T^2 - 2T + 1 + T^2(1-c)^2 + 2T(1-c) - 2T^2(1-c) + (1-c)^{2T} - 2(1-c)^T + 2T(1-c)^T - 2T(1-c)^{T+}}{T^2c^4} \\
&= \frac{-2}{c^2} + \frac{1}{c^4} + \frac{(1-c)^2}{c^4} - \frac{2(1-c)}{c^4} + o(1) + O(T^{-1}) + O(T^{-2}) + o(T^{-1}) + o(T^{-2}) \\
&= \frac{-1}{c^2} + o(1)
\end{aligned}$$

By (B.2),

$$\begin{aligned}
tr(D_T' D_T^2) &= tr((D_1'D_1 - D_2'D_2)(D_1 - D_2)) \\
&= tr(D_1'D_1D_1) - tr(D_1'D_1D_2) - tr(D_2'D_2D_1) + tr(D_2'D_2D_2)
\end{aligned}$$

By SE-5.a [11], when $\alpha \in (0, \infty)$

$$tr[D_1^{'}(D_1D_1)] = \sum_{t=1}^{T-2}\sum_{j=1}^{t} j\theta_T^{2j-1} = \theta_T^{-3}\sum_{t=1}^{T-2}\sum_{j=2}^{t+1}(j-1)\theta_T^{2j} = \theta_T^{-3}\sum_{t=1}^{T-2}\frac{\theta_T^4-(t+1)\theta_T^{2(t+2)}+t\theta_T^{2(t+3)}}{(1-\theta_T^2)^2}$$

$$= \frac{(T-2)\theta_T}{(1-\theta_T^2)^2} - \frac{\theta_T^{-3}}{(1-\theta_T^2)^2}\left[\theta_T^2\sum_{t=2}^{T-1}(t-1)\theta_T^{2t} + \sum_{t=2}^{T-1}\theta_T^{2(t+1)}\right] + \frac{\theta_T^{-3}\theta_T^4}{(1-\theta_T^2)^2}\left[\sum_{t=2}^{T-1}(t-1)\theta_T^{2t}\right]$$

$$= \frac{(T-2)\theta_T}{(1-\theta_T^2)^2} - \frac{\theta_T^{-1}-\theta_T}{(1-\theta_T^2)^2}\left[\frac{\theta_T^4-(T-1)\theta_T^{2(T)}+(T-2)\theta_T^{2(T+1)}}{(1-\theta_T^2)^2}\right] - \frac{\theta^{-3}}{(1-\theta_T^2)^2}\theta_T^{2*3}\sum_{j=1}^{T-2}\theta_T^{2(T-2-j)}$$

$$= \frac{(T-2)\theta_T}{(1-\theta_T^2)^2} - \frac{\theta_T^{-1}-\theta_T}{(1-\theta_T^2)^2}\left[\frac{\theta_T^4-(T-1)\theta_T^{2(T)}+(T-2)\theta_T^{2(T+1)}}{(1-\theta_T^2)^2}\right] - \frac{\theta_T^3(1-\theta_T^2)^{-1}(1-\theta_T^{2(T-2)})}{(1-\theta_T^2)^2}$$

$$= \left\{\frac{q(T)^2}{4}E_2\left(\frac{2}{q(T)},1\right)(T-2)\theta_T - \frac{q(T)^4}{16}E_4\left(\frac{2}{q(T)},3\right)(\theta_T^3-\theta_T^5) - \frac{q(T)^3}{8}E_3\left(\frac{2}{q(T)},2\right)\theta_T^3\right\}$$

$$+ \frac{q(T)^4}{16}E_4\left(\frac{2}{q(T)},3\right)\left[(T-1)(\theta_T^{2T-1}-\theta_T^{2T+1}) - (T-2)(\theta_T^{2T+1}-\theta_T^{2T+3})\right] + \frac{q(T)^3}{8}E_3\left(\frac{2}{q(T)},\right)\theta_T^{2T-1}$$

$$= \left\{\frac{q(T)^2}{4}\left(1+\frac{2}{q(T)}\right)(T-2)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!q(T)^n}\right) - \frac{q(T)^4}{16}\left(1+\frac{4}{q(T)}+\frac{11}{6}\frac{4}{q(T)^2}+\frac{8}{q(T)^3}\right)\left(\sum_{n=0}^{\infty}\frac{(-3)^n-(-5)^n}{n!q(T)^n}\right)\right.$$

$$\left. -\frac{q(T)^3}{8}\left(1+\frac{3}{q(T)}+\frac{4}{q(T)^2}\right)\left(\sum_{n=0}^{\infty}\frac{(-3)^n}{n!q(T)^n}\right)\right\}$$

$$+ \frac{q(T)^4}{16}\left(1+\frac{4}{q(T)}+\frac{11}{6}\frac{4}{q(T)^2}+\frac{8}{q(T)^3}\right)\left[(T-1)\left(\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}(-2T)^n\frac{1-(-1)^{j-n}}{n!(j-n)!q(T)^j}\right)\right.$$

$$\left. -(T-2)\left(\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}(-2T)^n\frac{(-1)^{j-n}-(-3)^{j-n}}{n!(j-n)!q(T)^j}\right)\right] + \frac{q(T)^3}{8}\left(1+\frac{3}{q(T)}+\frac{4}{q(T)^2}\right)\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}\frac{(-2T)^n}{n!(j-n)!q(T)^j}$$

$$= \left[q(T)^2T + q(T)T + q(T)^4 + q(T)^3 + q(T)^2 + q(T)\right]*0 + \sum_{n=1}^{\infty}\frac{T^n}{q(T)^{n-5}}*0 + \sum_{n=0}^{\infty}\frac{T^n}{q(T)^{n-4}}*0$$

$$+ \sum_{n=3}^{\infty}\frac{T^n}{q(T)^{n-3}}*\left[\frac{(-2)^n}{4(n!)}+\frac{(-2)^{n-1}}{4(n-1)!}*1_{n\geq 1}\right] + \sum_{n=2}^{\infty}\frac{T^n}{q(T)^{n-2}}*\left[\frac{(-2)^n}{2(n!)}+\frac{3(-2)^{n-1}}{4(n-1)!}1_{n\geq 1}\right] + o(T^2)$$

when $\alpha = 0$,

$$tr[D_1^{'}(D_1D_1)] = \frac{(T-2)\theta_T}{(1-\theta_T^2)^2} - \frac{\theta_T^{-1}-\theta_T}{(1-\theta_T^2)^2}\left[\frac{\theta_T^4-(T-1)\theta_T^{2(T)}+(T-2)\theta_T^{2(T+1)}}{(1-\theta_T^2)^2}\right] - \frac{\theta_T^3(1-\theta_T^2)^{-1}(1-\theta_T^{2(T-2)})}{(1-\theta_T^2)^2}$$

$$= \frac{(T-2)(1-c)}{(1-(1-c)^2)^2} - \frac{(1-c)^{-1}-(1-c)}{(1-(1-c)^2)^2}\left[\frac{(1-c)^4-(T-1)(1-c)^{2T}+(T-2)(1-c)^{2(T+1)}}{(1-(1-c)^2)^2}\right] -$$

$$\frac{(1-c)^3(1-(1-c)^{2(T-2)})}{(1-(1-c)^2)^3}$$

$$= \frac{(T-2)(1-c)}{(1-(1-c)^2)^2} - \frac{(1-c)^{-1}-(1-c)}{(1-(1-c)^2)^2}\left[\frac{(1-c)^4}{(1-(1-c)^2)^2}+o(T)\right] - \frac{(1-c)^3}{(1-(1-c)^2)^3} + o(1)$$

When $\alpha \in (0, \infty)$,

$$tr[D_1^{'}(D_1D_2)] = \frac{1}{T}\sum_{t=2}^{T} M_{T-t}\sum_{j=t-2}^{T-2}\theta_T^{j-t+2}M_j = \frac{1}{T}\sum_{t=2}^{T}\frac{1-\theta_T^{T-t+1}}{1-\theta_T}\sum_{j=t-2}^{T-2}\theta_T^{j-t+2}\frac{1-\theta_T^{j+1}}{1-\theta_T}$$

$$= \frac{1}{T(1-\theta_T)^2}\left[\sum_{t=2}^{T}(1-\theta_T^{T-t+1})\frac{1-\theta_T^{T-t+1}}{1-\theta_T} - \sum_{t=2}^{T}(1-\theta_T^{T-t+1})\frac{\theta_T^{t-1}(1-\theta_T^{2(T-t+1)})}{1-\theta_T^2}\right]$$

$$= \frac{T-1-2\theta_T\sum_{t=1}^{T-1}\theta_T^{T-t-1}+\theta^2\sum_{t=1}^{T-1}\theta_T^{2(T-t-1)}}{T(1-\theta_T)^3} - \frac{\sum_{t=2}^{T}\theta_T^{t-1}(1-\theta_T^{T-t+1}-\theta_T^{2(T-t+1)}+\theta_T^{3(T-t+1)})}{T(1-\theta_T)^2(1-\theta_T^2)}$$

$$= \frac{T-1}{T(1-\theta_T)^3} - \frac{2\theta_T\sum_{j=1}^{T-1}\theta^{T-1-j}}{T(1-\theta)^3} + \frac{\theta_T^2\sum_{j=1}^{T-1}\theta_T^{2(T-1-j)}}{T(1-\theta)^3} - \frac{\theta_T\sum_{j=1}^{T-1}\theta_T^{T-1-j}}{T(1-\theta_T)^2(1-\theta_T^2)}$$

$$+ \frac{(T-1)\theta_T^T}{T(1-\theta)^2(1-\theta_T^2)} + \frac{\theta_T^{T+1}\sum_{j=1}^{T-1}\theta_T^{T-1-j}}{T(1-\theta_T)^2(1-\theta_T^2)} - \frac{\theta_T^{T+2}\sum_{j=1}^{T-1}\theta_T^{2(T-1-j)}}{T(1-\theta_T)^2(1-\theta_T^2)}$$

$$= \frac{T-1}{T(1-\theta_T)^3} - \frac{(2\theta_T-2\theta_T^T)}{T(1-\theta_T)^4} + \frac{(\theta_T^2-\theta_T^{2T})-(\theta_T-\theta_T^T)}{T(1-\theta_T)^3(1-\theta_T^2)}$$

$$+ \frac{(T-1)\theta_T^T}{T(1-\theta_T)^2(1-\theta_T^2)} + \frac{(\theta_T^{T+1}-\theta_T^{2T})}{T(1-\theta_T)^3(1-\theta_T^2)} - \frac{(\theta_T^{T+2}-\theta_T^{3T})}{T(1-\theta_T)^2(1-\theta_T^2)^2}$$

$$= \left\{\frac{T-1}{T(1-\theta_T)^3} - \frac{2\theta_T}{T(1-\theta_T)^4} + \frac{\theta_T^2-\theta_T}{T(1-\theta_T)^3(1-\theta_T^2)}\right\}$$

$$+ \frac{2\theta_T^T}{T(1-\theta_T)^4} + \frac{\theta_T^T-2\theta_T^{2T}+\theta_T^{T+1}}{T(1-\theta_T)^3(1-\theta_T^2)} + \frac{(T-1)\theta_T^T}{T(1-\theta_T)^2(1-\theta_T^2)} - \frac{(\theta_T^{T+2}-\theta_T^{3T})}{T(1-\theta_T)^2(1-\theta_T^2)^2}$$

$$= \Bigg\{q(T)^3\left(1+\frac{3}{2}\frac{1}{q(T)}+\frac{1}{q(T)^2}\right)\frac{T-1}{T} - \frac{q(T)^4}{T}\left(1+2\frac{1}{q(T)}+\frac{11}{6}\frac{1}{q(T)^2}+\frac{1}{q(T)^3}\right)\sum_{n=0}^{\infty}\frac{2(-1)^n}{n!q(T)^n}+$$

$$\frac{q(T)^4}{2T}\left(1+\frac{5}{2q(T)}+\frac{17}{6q(T)^2}+\frac{15}{8q(T)^3}\right)\sum_{n=0}^{\infty}\frac{(-2)^n-(-1)^n}{n!q(T)^n}\Bigg\}$$

$$+ \frac{q(T)^4}{T}\left(1+2\frac{1}{q(T)}+\frac{11}{6}\frac{1}{q(T)^2}+\frac{1}{q(T)^3}\right)\sum_{n=0}^{\infty}\frac{2(-T)^n}{n!q(T)^n}$$

$$+ \frac{q(T)^4}{2T}\left(1+\frac{5}{2q(T)}+\frac{17}{6q(T)^2}+\frac{15}{8q(T)^3}\right)\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}(-T)^n\frac{0^{j-n}-2^{n+1}0^{j-n}+(-1)^{j-n}}{n!(j-n)!q(T)^j}$$

$$+ \frac{q(T)^3}{2}\left(1+\frac{2}{q(T)}+\frac{7}{4q(T)^2}\right)\frac{T-1}{T}\sum_{n=0}^{\infty}\frac{(-T)^n}{n!q(T)^n}$$

$$- \frac{q(T)^4}{4T}\left(1+\frac{3}{q(T)}+\frac{49}{12q(T)^2}+\frac{13}{4q(T)^3}\right)\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}(-T)^n\frac{(-2)^{j-n}-3^n0^{j-n}}{n!j-n!q(T)^j}$$

$$= \left[q(T)^3+q(T)^2+q(T)+\frac{q(T)^4+q(T)^3+q(T)^2+q(T)}{T}\right]*0$$

$$+ \sum_{n=3}^{\infty}\frac{T^n}{q(T)^{n-3}}*\left[\frac{(-1)^n/2}{n!}+\frac{(-3)^{n+1}/4+11(-1)^{n+1}/4-(-2)^{n+1}}{(n+1)!}\right]$$

$$+ \sum_{n=2}^{\infty}\frac{T^n}{q(T)^{n-2}}*\left[\frac{(-1)^{n+1}}{(n+1)!}(\frac{21}{4}-5*2^n+\frac{3^{n+2}}{4})+\frac{(-1)^n}{n!}\right]$$

When $\alpha = 0$,

$$
\begin{aligned}
tr[D_1^{'}(D_1D_2)] &= \left\{ \frac{T-1}{T(1-\theta_T)^3} - \frac{2\theta_T}{T(1-\theta_T)^4} + \frac{\theta_T^2 - \theta_T}{T(1-\theta_T)^3(1-\theta_T^2)} \right\} \\
&+ \frac{2\theta_T^T}{T(1-\theta_T)^4} + \frac{\theta_T^T - 2\theta_T^{2T} + \theta_T^{T+1}}{T(1-\theta_T)^3(1-\theta_T^2)} + \frac{(T-1)\theta_T^T}{T(1-\theta_T)^2(1-\theta_T^2)} - \frac{(\theta_T^{T+2} - \theta_T^{3T})}{T(1-\theta_T)^2(1-\theta_T^2)^2} \\
&= \left\{ \frac{T-1}{Tc^3} - \frac{2(1-c)}{Tc^4} + \frac{(1-c)^2 - (1-c)}{Tc^3(1-(1-c)^2)} \right\} \\
&+ \frac{2(1-c)^T}{Tc^4} + \frac{(1-c)^T - 2(1-c)^{2T} + (1-c)^{T+1}}{Tc^3(1-(1-c)^2)} + \frac{(T-1)(1-c)^T}{Tc^2(1-(1-c)^2)} - \frac{((1-c)^{T+2} - (1-c)^{3T})}{Tc^2(1-(1-c)^2)^2} \\
&= \left\{ \frac{1}{c^3} + O(T^{-1}) \right\} + o(T^{-1}) + o(1)
\end{aligned}
$$

When $\alpha \in (0, \infty)$

$$tr(D_2'D_2D_1) = \frac{1}{T}\sum_{t=2}^{T-1} M_{T-t} \sum_{j=t+1}^{T} \theta_T^{j-t-1} M_{T-j}$$

$$= \frac{1}{T}\sum_{t=2}^{T-1} \frac{1-\theta_T^{T-t+1}}{1-\theta_T} \sum_{j=t+1}^{T} \theta_T^{j-t-1} \frac{1-\theta_T^{T-j+1}}{1-\theta_T}$$

$$= \frac{1}{T(1-\theta_T)^2}\left[\sum_{t=2}^{T-1}(1-\theta_T^{T-t+1})\frac{1-\theta_T^{T-t}}{1-\theta_T} - \sum_{t=2}^{T-1}(1-\theta_T^{T-t+1})\theta_T^{T-t}(T-t)\right]$$

$$= \frac{T-2}{T(1-\theta_T)^3} - \frac{\theta_T(1-\theta_T^{T-2})}{T(1-\theta_T)^4} - \frac{\theta_T^2(1-\theta_T^{T-2})}{T(1-\theta_T)^4} + \frac{\theta_T^3(1-\theta_T^{2(T-2)})}{T(1-\theta_T)^3(1-\theta^2)}$$

$$- \frac{\sum_{t=2}^{T-1}\theta_T^{T-t}(T-t)}{T(1-\theta_T)^2} + \frac{\sum_{t=2}^{T-1}\theta_T^{2(T-t)+1}(T-t)}{T(1-\theta_T)^2}$$

$$= \frac{T-2}{T(1-\theta_T)^3} - \frac{2\theta_T - T\theta_T^{T-1} + \theta_T^2 + (T-3)\theta_T^T}{T(1-\theta_T)^4} + \frac{(\theta_T^3 - \theta_T^{(2T-1)})}{T(1-\theta_T)^3(1-\theta^2)} + \frac{\theta_T^3 - (T-1)\theta_T^{2T-1} + (T-2)\theta_T^{2T+1}}{T(1-\theta_T)^2(1-\theta_T^2)^2}$$

$$= \frac{q(T)^3 E_3\left(\frac{1}{q(T)}, 2\right)}{T}(T-2) - \frac{q(T)^4 E_4\left(\frac{1}{q(T)}, 3\right)}{T}(2\theta_T + \theta_T^2) - q(T)^4 E_4\left(\frac{1}{q(T)}, 3\right)\left[\theta_T^T - \theta_T^{T-1} - \frac{3}{T}\theta_T^T\right]$$

$$+ \frac{q(T)^4 E_3\left(\frac{1}{q(T)}, 3\right) E_1\left(\frac{2}{q(T)}, 3\right)}{2T}\left[\theta_T^3 - \theta_T^{2T-1}\right] + \frac{q(T)^4 E_2\left(\frac{1}{q(T)}, 3\right) E_2\left(\frac{2}{q(T)}, 3\right)}{4T}\left[\theta_T^3 + \theta_T^{2T-1} - 2\theta_T^{2T+1}\right]$$

$$+ \frac{q(T)^4 E_2\left(\frac{1}{q(T)}, 3\right) E_2\left(\frac{2}{q(T)}, 3\right)}{4}\left[-\theta_T^{2T-1} + \theta_T^{2T+1}\right]$$

$$= \Bigg\{ q(T)^3\left(1 + \frac{3}{2}\frac{1}{q(T)} + \frac{1}{q(T)^2}\right)\frac{T-2}{T} - \frac{q(T)^4}{T}\left(1 + \frac{2}{q(T)} + \frac{11}{6}\frac{1}{q(T)^2} + \frac{1}{q(T)^3}\right)\sum_{n=0}^{\infty}\frac{2(-1)^n + (-2)^n}{n!q(T)^n} +$$

$$\frac{q(T)^4}{2T}\left(1 + \frac{5}{2q(T)} + \frac{17}{6q(T)^2} + \frac{15}{8q(T)^3}\right)\sum_{n=0}^{\infty}\frac{(-3)^n}{n!q(T)^n} + \frac{q(T)^4}{4T}\left(1 + \frac{3}{q(T)} + \frac{49}{12q(T)^2} + \frac{13}{4q(T)^3}\right)\sum_{n=0}^{\infty}\frac{(-3)^n}{n!q(T)^n}\Bigg\}$$

$$- q(T)^4\left(1 + \frac{2}{q(T)} + \frac{11}{6q(T)^2} + \frac{1}{q(T)^3}\right)\sum_{n=0}^{\infty}(-T)^n\left[\sum_{j=n}^{\infty}\frac{0^{j-n} - 1}{n!(j-n)!q(T)^j}\right]$$

$$+ \frac{3q(T)^4}{T}\left(1 + \frac{2}{q(T)} + \frac{11}{6q(T)^2} + \frac{1}{q(T)^3}\right)\left[\sum_{n=0}^{\infty}\frac{(-T)^n}{n!q(T)^n}\right]$$

$$- \frac{q(T)^4}{2T}\left(1 + \frac{5}{2q(T)} + \frac{17}{6q(T)^2} + \frac{15}{8q(T)^3}\right)\sum_{n=0}^{\infty}(-2T)^n\sum_{j=n}^{\infty}\frac{1}{n!(j-n)!q(T)^j}$$

$$+ \frac{q(T)^4}{4T}\left(1 + \frac{3}{q(T)} + \frac{49}{12q(T)^2} + \frac{13}{4q(T)^3}\right)\sum_{n=0}^{\infty}(-2T)^n\sum_{j=n}^{\infty}\frac{1 - 2(-1)^{j-n}}{n!(j-n)!q(T)^j}$$

$$+ \frac{q(T)^4}{4}\left(1 + \frac{3}{q(T)} + \frac{49}{12q(T)^2} + \frac{13}{4q(T)^3}\right)\sum_{n=0}^{\infty}(-2T)^n\sum_{j=n}^{\infty}\frac{(-1)^{j-n} - 1}{n!(j-n)!q(T)^j}$$

$$= \left[q(T)^3 + q(T)^2 + q(T) + \frac{q(T)^4 + q(T)^3 + q(T)^2 + q(T)}{T} \right] * 0$$

$$+ \sum_{n=3}^{\infty} \frac{T^n}{q(T)^{n-3}} * \left[\frac{(-1)^n - (-2)^n/2}{n!} + \frac{3(-1)^{n+1} - (-2)^{n+1}/2 - (-2)^{n+1}/4}{(n+1)!} \right] +$$

$$\sum_{n=2}^{\infty} \frac{T^n}{q(T)^{n-2}} * \left[\frac{2.5(-1)^n - \frac{3}{2}(-2)^n}{n!} + \frac{6(-1)^{n+1} - \frac{7}{4}(-2)^{n+1}}{(n+1)!} \right]$$

$When\ \alpha = 0,$

$$tr(D_2' D_2 D_1)$$

$$= \frac{T-2}{T(1-\theta_T)^3} - \frac{2\theta_T - T\theta_T^{T-1} + \theta_T^2 + (T-3)\theta_T^T}{T(1-\theta_T)^4} + \frac{(\theta_T^3 - \theta_T^{(2T-1)})}{T(1-\theta_T)^3(1-\theta^2)} + \frac{\theta_T^3 - (T-1)\theta_T^{2T-1} + (T-2)\theta_T^{2T+1}}{T(1-\theta_T)^2(1-\theta_T^2)^2}$$

$$= \frac{1}{c^3} + O(T^{-1}) + o(1) + o(T^{-1})$$

When $\alpha \in (0,\infty)$ *By SE-1[11] with* $d=1, b=2$, *SE-2[11] with* $b=1, g=1$,

$$tr(D_2'D_2D_2) = \left[\frac{1}{T^2}\sum_{j=2}^{T} M_{T-j}\sum_{j=2}^{T} M_{T-j}^2\right] = \frac{1}{T^2}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\theta_T^{t-1-j}\sum_{t=2}^{T}(\sum_{j=1}^{t-1}\theta_T^{t-1-j})^2$$

$$= \frac{T-1-\theta_T(1-\theta_T)^{-1}(1-\theta_T^{T-1})}{T^2(1-\theta_T)}\frac{T-1-2\theta_T(1-\theta_T)^{-1}(1-\theta_T^{T-1})+\theta_T^2(1-\theta_T^2)^{-1}(1-\theta_T^{2(T-1)})}{(1-\theta_T)}$$

$$= \left[\frac{T-1}{T^2(1-\theta_T)} - \frac{(\theta_T-\theta_T^T)}{T^2(1-\theta_T)^2}\right]\left[\frac{T-1}{(1-\theta_T)^2} - \frac{2(\theta_T-\theta_T^T)}{(1-\theta_T)^3} + \frac{(\theta_T^2-\theta_T^{2T})}{(1-\theta_T)^2(1-\theta_T^2)}\right]$$

$$= \frac{(T-1)^2}{T^2(1-\theta_T)^3} - \frac{3(T-1)(\theta_T-\theta_T^T)}{T^2(1-\theta_T)^4} + \frac{(T-1)(\theta_T^2-\theta_T^{2T})}{T^2(1-\theta_T)^3(1-\theta_T^2)} + \frac{2(\theta_T-\theta_T^T)^2}{T^2(1-\theta_T)^5} - \frac{(\theta_T-\theta_T^T)(\theta_T^2-\theta_T^{2T})}{T^2(1-\theta_T)^4(1-\theta_T^2)}$$

$$= \left\{\frac{(T-1)^2}{T^2(1-\theta_T)^3} - \frac{3(T-1)\theta_T}{T^2(1-\theta_T)^4} + \frac{(T-1)\theta_T^2}{T^2(1-\theta_T)^3(1-\theta_T^2)} + \frac{2\theta_T^2}{T^2(1-\theta_T)^5} - \frac{\theta_T^3}{T^2(1-\theta_T)^4(1-\theta_T^2)}\right\}$$

$$+ \frac{3(T-1)\theta_T^T}{T^2(1-\theta_T)^4} - \frac{(T-1)\theta_T^{2T}}{T^2(1-\theta_T)^3(1-\theta_T^2)} + \frac{(-4\theta_T^{T+1}+2\theta_T^{2T})}{T^2(1-\theta_T)^5} + \frac{(\theta_T^{2T+1}+\theta_T^{T+2}-\theta_T^{3T})}{T^2(1-\theta_T)^4(1-\theta_T^2)}$$

$$= \left\{O(\frac{q(T)^5}{T^2})\right\} + \frac{3q(T)^4(T-1)}{T^2}\left(1+2\frac{1}{q(T)}+\frac{11}{6}\frac{1}{q(T)^2}+\frac{1}{q(T)^3}\right)\sum_{n=0}^{\infty}\frac{(-T)^n}{n!q(T)^n}$$

$$- \frac{q(T)^4(T-1)}{2T^2}\left(1+\frac{5}{2q(T)}+\frac{17}{6q(T)^2}+\frac{15}{8q(T)^3}\right)\sum_{n=0}^{\infty}\frac{(-2T)^n}{n!q(T)^n}$$

$$+ \frac{q(T)^5}{T^2}\left(1+\frac{5}{2}\frac{1}{q(T)}+\frac{35}{12}\frac{1}{q(T)^2}+\frac{25}{12}\frac{1}{q(T)^3}\right)\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}(-T)^n\frac{2^{n+1}0^{j-n}-4(-1)^{j-n}}{n!j-n!q(T)^j}$$

$$+ \frac{q(T)^5}{2T^2}\left(1+\frac{3}{q(T)}+\frac{25}{6q(T)^2}+\frac{7}{2q(T)^3}\right)\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}(-T)^n\frac{(2)^n(-1)^{j-n}+(-2)^{j-n}-3^n0^{j-n}}{n!j-n!q(T)^j}$$

$$= \left[\frac{q(T)^5+q(T)^4+q(T)^3+q(T)^2+q(T)}{T^2} + \frac{q(T)^4+q(T)^3+q(T)^2+q(T)}{T} + q(T)^3+q(T)^2+q(T)\right]*0$$

$$+ \sum_{n=3}^{\infty}\frac{T^n}{q(T)^{n-3}}\left[\frac{3(-1)^{n+1}-(-2)^{n+1}/2}{(n+1)!} + \frac{2.5(-2)^{n+2}-3.5(-1)^{n+2}-(-3)^{n+2}/2}{(n+2)!}\right]$$

$$+ \sum_{n=2}^{\infty}\frac{T^n}{q(T)^{n-2}}\left[\frac{6(-1)^{n+1}-5(-2)^{n+1}/4}{(n+1)!} + \frac{(-2)^{n+2}6-8.5(-1)^{n+2}-(-3)^{n+2}1.5+(-2)^{n+2}/2}{(n+2)!}\right]$$

When $\alpha = 0$,

$$tr(D_2'D_2D_2) = \left\{\frac{1}{c^3} + O(T^{-1}) + O(T^{-2})\right\} + o(T^{-1}) + o(T^{-2})$$

So that when $\alpha \in (0, \infty)$,

$$
\begin{aligned}
&tr(D^{'}DD) = tr(D_1^{'}D_1D_1) - tr(D_1^{'}D_1D_2) - tr(D_2^{'}D_2D_1) + tr(D_2^{'}D_2D_2) \\
&= T^3 * 0 + \sum_{n=4}^{\infty} \frac{T^n}{q(T)^{n-3}} * \Bigg[\frac{(-2)^{n-1}/4}{(n-1)!} + \frac{0.75(-2)^n - 1.5(-1)^n}{n!} \\
&- \frac{(-3)^{n+1}/4 + 11(-1)^{n+1}/4 - 5(-2)^{n+1}/4}{(n+1)!} + \frac{2.5(-2)^{n+2} - 3.5(-1)^{n+2} - (-3)^{n+2}/2}{(n+2)!} \Bigg] \\
&- \frac{1}{12}T^2 + \sum_{n=3}^{\infty} \frac{T^n}{q(T)^{n-2}} * \Bigg[\frac{3(-2)^{n-1}/4}{(n-1)!} + \frac{-3.5(-1)^n + 2(-2)^n}{n!} - \frac{5.25(-1)^{n+1} - 3 * (-2)^{n+1} + 0.75(-3)^{n+1}}{(n+1)!} \\
&+ \frac{6.5(-2)^{n+2} - 8.5(-1)^{n+2} - (-3)^{n+2}1.5}{(n+2)!} \Bigg] + o(T^2)
\end{aligned}
$$

When $\alpha > 2$, $tr(D^{'}DD) = -\frac{1}{12}T^2 + o(T^2)$. *When* $\alpha = 2$, $tr(D^{'}DD) = \left(-\frac{1}{12} + \frac{c}{90}\right)T^2 + o(T^2)$. *When* $1 < \alpha < 2$, $tr(D^{'}DD) = o(T^3)$. *When* $\alpha = 1$, $tr(D^{'}DD) = O(T^3)$.

So that when $\alpha = 0$,

$$
\begin{aligned}
tr(D^{'}DD) &= tr(D_1^{'}D_1D_1) - tr(D_1^{'}D_1D_2) - tr(D_2^{'}D_2D_1) + tr(D_2^{'}D_2D_2) \\
&= \frac{T(1-c)}{(1-(1-c)^2)^2} + o(T)
\end{aligned}
$$

For $tr[(D_T^{'}D_T)^2]$,

$$
\begin{aligned}
tr[(D_T^{'}D_T)^2] &= tr[D_1^{'}D_1D_1^{'}D_1] - tr[D_1^{'}D_1D_1^{'}D_2] - tr[D_1^{'}D_2D_1^{'}D_1] + tr[D_1^{'}D_2D_1^{'}D_2] \\
&= tr[D_1^{'}D_1D_1^{'}D_1] - 2tr[D_1^{'}D_1D_1^{'}D_2] + tr[D_1^{'}D_2D_1^{'}D_2]
\end{aligned}
$$

When $\alpha \in (0, \infty)$,

$$tr[D_1 D_1^{'} D_1 D_1^{'}] = 2\sum_{t=0}^{T-2}\sum_{j=0}^{T-t-2}(\sum_{k=0}^{j}\theta_T^{2k+t})^2 - \sum_{j=0}^{T-2}(\sum_{k=0}^{j}\theta_T^{2k})^2$$

$$= 2\sum_{t=0}^{T-2}\theta_T^{2t}\sum_{j=1}^{T-t-1}(\sum_{k=1}^{j}\theta_T^{2(j-k)})^2 - \sum_{j=1}^{T-1}(\sum_{k=1}^{j}\theta_T^{2(j-k)})^2$$

$$= \left[2\sum_{t=0}^{T-2}\frac{(T-t-1)\theta_T^{2t}}{\left(1-\theta_T^2\right)^2} - \frac{2\theta_T^{2(t+1)} - 2\theta_T^{2T}}{\left(1-\theta_T^2\right)^3} + \frac{\theta_T^{4+2t} - \theta_T^{4T-2t}}{(1-\theta_T^2)^2(1-\theta_T^4)}\right] - \frac{(T-1)}{\left(1-\theta_T^2\right)^2} + \frac{2\theta_T^2 - 2\theta_T^{2T}}{\left(1-\theta_T^2\right)^3} - \frac{\theta_T^4 - \theta_T^{4T}}{(1-\theta_T^2)^2(1-\theta_T^4)}$$

$$= 2T\frac{1-\theta_T^{2(T-1)}}{\left(1-\theta_T^2\right)^3} - 2\sum_{t=2}^{T}\frac{(t-2)\theta_T^{2(t-2)} + \theta_T^{2(t-2)}}{\left(1-\theta_T^2\right)^2} - \frac{4\theta_T^2 - 4\theta_T^{2T}}{\left(1-\theta_T^2\right)^4} + \frac{(T-1)4\theta_T^{2T}}{\left(1-\theta_T^2\right)^3} + \frac{2\theta_T^4 - 2\theta_T^{2(T+1)}}{(1-\theta_T^2)^3(1-\theta_T^4)}$$

$$- \frac{2\theta_T^{4T} - 2\theta_T^{(2T+2)}}{(1-\theta_T^{-2})(1-\theta_T^2)^2(1-\theta_T^4)} - \frac{(T-1)}{\left(1-\theta_T^2\right)^2} + \frac{2\theta_T^2 - 2\theta_T^{2T}}{\left(1-\theta_T^2\right)^3} - \frac{\theta_T^4 - \theta_T^{4T}}{(1-\theta_T^2)^2(1-\theta_T^4)}$$

$$= 2T\frac{1-\theta_T^{2(T-1)}}{\left(1-\theta_T^2\right)^3} - 2\frac{\theta_T^2 - (T-1)\theta_T^{2T-2} + (T-2)\theta_T^{2T}}{\left(1-\theta_T^2\right)^4} - 2\frac{1-\theta_T^{2(T-1)}}{\left(1-\theta_T^3\right)^2} - \frac{4\theta_T^2 - 4\theta_T^{2T}}{\left(1-\theta_T^2\right)^4} + \frac{(T-1)4\theta_T^{2T}}{\left(1-\theta_T^2\right)^3}$$

$$+ \frac{2\theta_T^4 - 2\theta_T^{2(T+1)}}{(1-\theta_T^2)^3(1-\theta_T^4)} - \frac{2\theta_T^{4T} - 2\theta_T^{(2T+2)}}{(1-\theta_T^{-2})(1-\theta_T^2)^2(1-\theta_T^4)} - \frac{(T-1)}{\left(1-\theta_T^2\right)^2} + \frac{2\theta_T^2 - 2\theta_T^{2T}}{\left(1-\theta_T^2\right)^3} - \frac{\theta_T^4 - \theta_T^{4T}}{(1-\theta_T^2)^2(1-\theta_T^4)}$$

$$= \frac{2T - (2T-2)\theta_T^{2(T-1)} + (4T-6)\theta_T^{2T} + 2\theta_T^2 - 2}{\left(1-\theta_T^2\right)^3} - \frac{6\theta_T^2 - 2(T-1)\theta_T^{2(T-1)} + (2T-8)\theta_T^{2T}}{\left(1-\theta_T^2\right)^4}$$

$$+ \frac{2\theta_T^4 - 2\theta_T^{2(T+1)}}{(1-\theta_T^2)^3(1-\theta_T^4)} - \frac{2\theta_T^{4T} - 2\theta_T^{(2T+2)}}{(1-\theta_T^{-2})(1-\theta_T^2)^2(1-\theta_T^4)} - \frac{(T-1)}{\left(1-\theta_T^2\right)^2} - \frac{\theta_T^4 - \theta_T^{4T}}{(1-\theta_T^2)^2(1-\theta_T^4)}$$

$$= \left\{\frac{2T + 2\theta_T^2 - 2}{\left(1-\theta_T^2\right)^3} - \frac{6\theta_T^2}{\left(1-\theta_T^2\right)^4} + \frac{2\theta_T^4}{(1-\theta_T^2)^3(1-\theta_T^4)} - \frac{(T-1)}{\left(1-\theta_T^2\right)^2} - \frac{\theta_T^4}{(1-\theta_T^2)^2(1-\theta_T^4)}\right\}$$

$$+ \frac{-(2T-2)\theta_T^{2(T-1)} + (4T-6)\theta_T^{2T}}{\left(\frac{q(T)^3}{8}(1+O(\frac{1}{q(T)}))\right)^{-1}} - \frac{-2(T-1)\theta_T^{2(T-1)} + (2T-8)\theta_T^{2T}}{\left(\frac{q(T)^4}{16}(1+\frac{4}{q(T)}+O(\frac{1}{q(T)^2}))\right)^{-1}}$$

$$+ \frac{-2\theta_T^{2(T+1)} + 2\theta_T^{4T+2} - 2\theta_T^{(2T+4)}}{\left(\frac{q(T)^4}{32}(1+O(\frac{1}{q(T)}))\right)^{-1}} - \frac{-\theta_T^{4T}}{\left(\frac{q(T)^3}{16}(1+O(\frac{1}{q(T)}))\right)^{-1}}$$

$$= \left\{Tq(T)^3 + Tq(T)^2 + Tq(T) + q(T)^4 + q(T)^3 + q(T)^2 + q(T)\right\} * 0$$

$$+ \sum_{n=1}^{\infty}\frac{T^n}{q(T)^{n-5}} * 0 + \sum_{n=4}^{\infty}\frac{T^n}{q(T)^{n-4}}\left(\frac{-(-2)^{n-2}}{n-1!}1_{n\geq 1} + \frac{(-2)^{n-2}}{n!} + \frac{(-4)^{n-2}}{n!}\right) + o(T^4)$$

When $\alpha = 0$,

$$tr[D_1 D_1^{'} D_1 D_1^{'}] = \frac{2T}{(1-(1-c)^2)^3} + o(T) - \frac{T}{(1-(1-c)^2)^2}$$

When $\alpha \in (0, \infty)$

$$
\begin{aligned}
&tr[D_1 D_1^{'} D_2 D_1^{'}] = tr[D_1 D_2^{'} D_2 D_1^{'}] \\
&= \frac{1}{T(1-\theta_T)^2} \sum_{t=0}^{T-2} \left(\frac{1+\theta_T^{2(t+1)} - 2\theta_T^{t+1}}{(1-\theta_T)^2} + \frac{\theta_T^{2T+2t+2} + \theta_T^{2T-2t-2} - 2\theta_T^{2T}}{(\theta_T^2-1)^2} + 2\frac{\theta_T^{T+t+1} - \theta_T^{T-t-1} - \theta_T^{T+2t+2} + \theta_T^{T}}{(1-\theta_T)(1-\theta_T^2)} \right) \\
&= \frac{T-1}{T(1-\theta_T)^4} + \frac{\theta_T^2 - \theta_T^{2T}}{T(1-\theta_T)^4(1-\theta_T^2)} - \frac{2\theta_T - 2\theta_T^T}{T(1-\theta_T)^5} + \frac{\theta_T^{2T+2} - \theta_T^{4T} - \theta_T^{2T} + \theta_T^2}{T(1-\theta_T)^2(1-\theta_T^2)^3} - \frac{2(T-1)\theta_T^{2T}}{T(1-\theta_T)^2(1-\theta_T^2)^2} \\
&+ \frac{2\theta_T^{T+1} - 2\theta_T^{2T} + 2\theta_T^T - 2\theta_T}{T(1-\theta_T)^4(1-\theta_T^2)} - \frac{2\theta_T^{T+2} - 2\theta_T^{3T}}{T(1-\theta_T)^3(1-\theta_T^2)^2} + \frac{2(T-1)\theta_T^T}{T(1-\theta_T)^3(1-\theta_T^2)} \\
&= \left\{ \frac{T-1}{T(1-\theta_T)^4} + \frac{\theta_T^2}{T(1-\theta_T)^4(1-\theta_T^2)} - \frac{2\theta_T}{T(1-\theta_T)^5} + \frac{\theta_T^2}{T(1-\theta_T)^2(1-\theta_T^2)^3} + \frac{-2\theta_T}{T(1-\theta_T)^4(1-\theta_T^2)} \right\} \\
&+ \frac{2\theta_T^{T+1} - 3\theta_T^{2T} + 2\theta_T^T}{T\left(\frac{q(T)^5}{2}(1+O(\frac{1}{q(T)}))\right)^{-1}} - \frac{-2\theta_T^T}{T\left(q(T)^5(1+O(\frac{1}{q(T)}))\right)^{-1}} + \frac{\theta_T^{2T+2} - \theta_T^{4T} - \theta_T^{2T}}{T\left(\frac{q(T)^5}{8}(1+O(\frac{1}{q(T)}))\right)^{-1}} \\
&- \frac{2(T-1)\theta_T^{2T}}{T\left(\frac{q(T)^4}{4}(1+O(\frac{1}{q(T)}))\right)^{-1}} - \frac{2\theta_T^{T+2} - 2\theta_T^{3T}}{T\left(\frac{q(T)^5}{4}(1+O(\frac{1}{q(T)}))\right)^{-1}} + \frac{2(T-1)\theta_T^T}{T\left(\frac{q(T)^4}{2}(1+O(\frac{1}{q(T)}))\right)^{-1}} \\
&= \left\{ \frac{q(T)^5 + q(T)^4 + q(T)^3 + q(T)^2 + q(T)}{T} + q(T)^4 + q(T)^3 + q(T)^2 + q(T) \right\} * 0 + \\
&\sum_{n=4}^{\infty} \frac{T^n}{q(T)^{n-4}} \left(\frac{(-1)^n - (-2)^n 0.5}{n!} + \frac{(-1)^{n+1}3.5 - (-2)^{n+1}1.5 - (-4)^{n+1}/8 + (-3)^{n+1}0.5}{(n+1)!} \right) + o(T^4)
\end{aligned}
$$

When $\alpha = 0$,

$$
tr[D_1 D_1^{'} D_2 D_1^{'}] = \frac{1}{c^4} + O(T^{-1}) + o(T^{-1}) + o(1)
$$

When $\alpha \in (0, \infty)$,

$$tr[D_1'D_2D_1'D_2] = tr[D_2'D_2D_2'D_2] = \frac{1}{T^2}\sum_{t=2}^{T}\sum_{j=2}^{T}(M_{T-t}M_{T-j})^2$$

$$= \frac{1}{T^2}\sum_{t=2}^{T}\sum_{j=2}^{T}\frac{(1-\theta_T^{T-j+1}-\theta_T^{T-t+1}+\theta_T^{2T-t-j+2})^2}{(1-\theta_T)^4}$$

$$= \frac{1}{T^2(1-\theta_T)^4}\sum_{t=2}^{T}\sum_{j=2}^{T}(1+\theta_T^{2T-2j+2}+\theta_T^{2T-2t+2}+\theta_T^{4T-2t-2j+4}-2\theta_T^{T-j+1}-2\theta_T^{T-t+1}+2\theta_T^{2T-t-j+2}+2\theta_T^{2T-j-t+2}$$
$$-2\theta_T^{3T-t-2j+3}-2\theta_T^{3T-2t-j+3})$$

$$= \frac{1}{T^2(1-\theta_T)^4}\sum_{t=2}^{T}\Bigg(T-1+\frac{\theta_T^2-\theta_T^{2T}}{1-\theta_T^2}+(T-1)\theta_T^{2T-2t+2}+\frac{\theta_T^{2T+4-2t}-\theta_T^{4T-2t+2}}{1-\theta_T^2}-2\frac{\theta_T-\theta_T^T}{1-\theta_T}-2(T-1)\theta_T^{T-t+1}$$
$$+4\frac{\theta_T^{T-t+2}-\theta_T^{2T-t+1}}{1-\theta_T}-2\frac{\theta_T^{T+3-t}-\theta_T^{3T+1-t}}{1-\theta_T^2}-2\frac{\theta_T^{2T-2t+3}-\theta_T^{3T-2t+2}}{1-\theta_T}\Bigg)$$

$$= \Bigg(\frac{(T-1)^2}{T^2(1-\theta_T)^4}+2(T-1)\frac{\theta_T^2-\theta_T^{2T}}{T^2(1-\theta_T)^4(1-\theta_T^2)}+\frac{\theta_T^4-\theta_T^{2T+2}-\theta_T^{2T+2}+\theta_T^{4T}}{T^2(1-\theta_T)^4(1-\theta_T^2)^2}-4(T-1)\frac{\theta_T-\theta_T^T}{T^2(1-\theta_T)^5}$$
$$+4\frac{\theta_T^2-2\theta_T^{T+1}+\theta_T^{2T}}{T^2(1-\theta_T)^6}-4\frac{\theta_T^3-\theta_T^{2T+1}-\theta_T^{T+2}+\theta_T^{3T}}{T^2(1-\theta_T)^5(1-\theta_T^2)}\Bigg)$$

$$= \Bigg(\frac{T^2-2T+1}{T^2(1-\theta_T)^4}+\frac{2(T-1)\theta_T^2}{T^2(1-\theta_T)^4(1-\theta_T^2)}+\frac{\theta_T^4}{T^2(1-\theta_T)^4(1-\theta_T^2)^2}-\frac{4(T-1)\theta_T}{T^2(1-\theta_T)^5}+\frac{4\theta_T^2}{T^2(1-\theta_T)^6}$$
$$-\frac{4\theta_T^3}{T^2(1-\theta_T)^5(1-\theta_T^2)}\Bigg)+\Bigg(2(T-1)\frac{-\theta_T^{2T}}{T^2(1-\theta_T)^4(1-\theta_T^2)}+\frac{-2\theta_T^{2T+2}+\theta_T^{4T}}{T^2(1-\theta_T)^4(1-\theta_T^2)^2}-4(T-1)\frac{-\theta_T^T}{T^2(1-\theta_T)^5}$$
$$+4\frac{-2\theta_T^{T+1}+\theta_T^{2T}}{T^2(1-\theta_T)^6}-4\frac{-\theta_T^{2T+1}-\theta_T^{T+2}+\theta_T^{3T}}{T^2(1-\theta_T)^5(1-\theta_T^2)}\Bigg)$$

$$= \Bigg(\frac{q(T)^6+q(T)^5+q(T)^4+q(T)^3+q(T)^2+q(T)}{T^2}+\frac{q(T)^5+q(T)^4+q(T)^3+q(T)^2+q(T)}{T}$$
$$+q(T)^4+q(T)^3+q(T)^2+q(T)\Big)*0$$
$$+\sum_{n=4}^{\infty}\frac{T^n}{q(T)^{n-4}}\left[\frac{4(-1)^{n+1}-(-2)^{n+1}}{(n+1)!}+\frac{(-2)^{n+1}-(-4)^{n+1}-6(-1)^{n+2}-3(-2)^{n+3}-2(-3)^{n+2}}{(n+2)!}\right]+o(T^4)$$

When $\alpha = 0$,

$$tr[D_1'D_2D_1'D_2] = \left(\frac{1}{c^4}+O(T^{-1})+O(T^{-2})\right)+\left(o(T^{-1})+o(T^{-2})\right)$$

Corollary 3. *Depending on the parameter* α, *we can divide Lemma 2 into four cases,* *(1) When $c \neq 0$, $\alpha > 1$, $q(T) = T^\alpha / c$,*

$$tr(D_T) = T\left(-\frac{1}{2}\right) + O\left(\frac{T^2}{q(T)}\right) + O(1);$$

$$tr(D_T D_T^{'}) = T^2\left(\frac{1}{6}\right) + O\left(\frac{T^3}{q(T)}\right) + O(T);$$

$$tr(D_T^2) = T^2\left(-\frac{1}{12}\right) + o(T^2);$$

$$tr[(D_T^{'} D_T)^2] = T^4\left(\frac{1}{90}\right) + o(T^4);$$

$$tr(D_T^{'} D_T^2) = \sum_{n=4}^{\infty} \frac{T^n}{q(T)^{n-3}} * \left[\frac{(-2)^{n-1}/4}{(n-1)!} + \frac{0.75(-2)^n - 1.5(-1)^n}{n!}\right.$$
$$\left. - \frac{(-3)^{n+1}/4 + 11(-1)^{n+1}/4 - 5(-2)^{n+1}/4}{(n+1)!} + \frac{2.5(-2)^{n+2} - 3.5(-1)^{n+2} - (-3)^{n+2}/2}{(n+2)!}\right]$$
$$- \frac{1}{12}T^2 + o(T^2);$$

(2) When $c \neq 0$, $\alpha = 1$, $q(T) = T^\alpha / c$,

$$tr(D_T) = -T\frac{e^{-c} + c - 1}{c^2} + O(1);$$

$$tr(D_T D_T^{'}) = T^2\left(-\frac{1 - e^{-2c} - 2c + 2c^2}{2c^3} + \frac{e^{-2c} - 1 + 2c}{4c^2} - \frac{2e^{-c} - 2 + 2c - c^2}{c^3}\right) + O(T);$$

$$tr(D_T^2) = T^2\left(\frac{-2e^{-c} + 2 - 2c}{c^2} + \frac{e^{-2c} - 2e^{-c} + 1 - c^2 + c^3}{c^4} - \frac{2e^{-c} - 2 + 2c - c^2}{c^3}\right) + O(T);$$

$$tr[(D_T^{'} D_T)^2] = T^4\left[\frac{e^{-2c} - 1 + 2c - 2c^2}{2c^3} + \frac{20e^{-2c} + 11 + 12c - 32c^2 + 32c^3 + e^{-4c} - 32e^{-c}}{16c^4}\right.$$
$$+ \frac{-12e^{-c} + 7 - 4c + 4c^3/3 - 2c^4 + 8e^{-2c} + e^{-4c} - 4e^{-3c}}{4c^5}$$
$$\left. + \frac{(e^{-4c}) - 24(e^{-c}) + 22(e^{-2c}) - 8(e^{-3c}) - (-9 + 4c^2 - (13/6)c^4 + 2c^5)}{4c^6}\right] + O(T^3);$$

$$tr(D_T^{'} D_T^2) = T^3 * \left[\frac{e^{-2c} - 1 + 2c - 2c^2}{4c^2} + \frac{3e^{-2c} + 3 - 3c^2 + 3c^3 - 6e^{-c}}{4c^3}\right.$$
$$\left. - \frac{e^{-3c} + 11e^{-c} - 5e^{-2c} - 7 + 4c - c^3/3 - c^4/2}{4c^4} + \frac{5e^{-2c} - 7e^{-c} - e^{-3c} + 3 - 2c^2 + c^3 + c^4/3 - 0.75c^5}{2c^5}\right]$$
$$+ O(T^2)$$

(3) When $c \neq 0$, $q(T) = T^{\alpha}/c$, $0 < \alpha < 1$

$$\begin{aligned}
tr(D_T) &= -\frac{T^{\alpha}}{c} + o(T^{\alpha}) \\
tr(D_T D_T^{'}) &= \frac{T^{1+\alpha}}{2c} + o(T^{1+\alpha}) \\
tr(D_T^2) &= -\frac{T^{2\alpha}}{c^2} + o(T^{2\alpha}) \\
tr[(D_T^{'} D_T)^2] &= \frac{T^{1+3\alpha}}{4c^3} + o(T^{1+3\alpha}) \\
tr(D_T^{'} D_T^2) &= \frac{T^{1+2\alpha}}{4c^2} + o(T^{1+2\alpha})
\end{aligned}$$

(4) When $c \neq 0$, $q(T) = T^{\alpha}/c$, $\alpha = 0$, result same as Lemma 2.(ii).

Proof 2. *The proof of case (3) can be derived by (B.1).*

Corollary 4. *For the term $tr(D_T^{'} D_T^2)$ when $\alpha > 1$ in Corollary 3.(1),*

(1) When $c \neq 0$, $\alpha > 2$, $q(T) = T^{\alpha}/c$,

$$tr(D_T^{'} D_T^2) = -\frac{1}{12} T^2 + o(T^2);$$

(2) When $c \neq 0$, $\alpha = 2$, $q(T) = T^{\alpha}/c$,

$$tr(D_T^{'} D_T^2) = (\frac{c}{90} - \frac{1}{12}) T^2 + o(T^2);$$

(3) When $c \neq 0$, $1 < \alpha < 2$, $q(T) = T^{\alpha}/c$,

$$tr(D_T^{'} D_T^2) = \frac{T^4}{T^{\alpha}} * \frac{c}{90} + O(\frac{T^5}{T^{2\alpha}});$$

Corollary 5 (The expression of $tr(G_T), tr(M_T^2), \frac{B_{N,T}(\theta_T,\sigma^2)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}}$).

$$\begin{aligned}
&G_T = D_T + \frac{3}{T+1} D_T^{\prime} D_T, \\
&tr(M_T^2) = \frac{1}{2}\left[tr(DD) + tr(D^{\prime}D) + \frac{12}{T+1} tr(D^{\prime}D^2) + \frac{18}{(T+1)^2} tr((D^{\prime}D)^2) \right], \\
&\frac{B_{N,T}(\theta_T,\sigma^2)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} = \frac{-3/(T+1)}{\sqrt{V(\theta_T,\sigma^2)}} + \frac{N\sigma^2 tr(G_T)}{\sqrt{N2\sigma^4 tr(M_T^2)}}.
\end{aligned}$$

(0) When $c = 0$*,*

$$tr(G_T) = 0,$$
$$tr(M_T^2) = \frac{1}{120}\frac{17T^3 - 37T^2 + 37T - 17}{T+1},$$
$$\frac{B_{N,T}(\theta_T, \sigma^2)}{\sqrt{V_{N,T}(\theta_T, \sigma^2)}} = \frac{-3/(T+1)}{\sqrt{V(\theta_T, \sigma^2)}} + 0.$$

(1) When $\alpha > 1$*,*

$$tr(G_T) = T * 0 + O\left(\frac{T^2}{T^\alpha}\right) + O(1),$$
$$tr(M_T^2) = \frac{1}{2}\frac{17T^2}{60} + o(T^2),$$
$$\frac{B_{N,T}(\theta_T, \sigma^2)}{\sqrt{V_{N,T}(\theta_T, \sigma^2)}} = \frac{-3/(T+1)}{\sqrt{V_{N,T}(\theta_T, \sigma^2)}} + O\left(\frac{\sqrt{N}}{T^{\alpha-1}}\right).$$

(2) When $\alpha = 1$*,*

$$tr(G_T) = TA_c + O(1),$$
$$tr(M_T^2) = T^2 B_c + o(T^2),$$
$$\frac{B_{N,T}(\theta_T, \sigma^2)}{\sqrt{V_{N,T}(\theta_T, \sigma^2)}} = \frac{-3/(T+1)}{\sqrt{V(\theta_T, \sigma^2)}} + \frac{\sqrt{N}A_c}{\sqrt{2B_c}} + o(\sqrt{N}).$$

where $A_c = \left[-\frac{e^{-c}+c-1}{c^2} + 3\left(-\frac{1-e^{-2c}-2c+2c^2}{2c^3} + \frac{e^{-2c}-1+2c}{4c^2} - \frac{2e^{-c}-2+2c-c^2}{c^3}\right)\right]$,

$$\begin{aligned}
B_c = \frac{1}{2}\Bigg[&\left(\frac{-2e^{-c}+2-2c}{c^2} + \frac{e^{-2c}-2e^{-c}+1-c^2+c^3}{c^4} - \frac{2e^{-c}-2+2c-c^2}{c^3}\right) \\
&+\left(-\frac{1-e^{-2c}-2c+2c^2}{2c^3} + \frac{e^{-2c}-1+2c}{4c^2} - \frac{2e^{-c}-2+2c-c^2}{c^3}\right) \\
&+12\Bigg(\frac{e^{-2c}-1+2c-2c^2}{4c^2} + \frac{3e^{-2c}+3-3c^2+3c^3-6e^{-c}}{4c^3} \\
&\quad - \frac{e^{-3c}+11e^{-c}-5e^{-2c}-7+4c-c^3/3-c^4/2}{4c^4} + \frac{5e^{-2c}-7e^{-c}-e^{-3c}+3-2c^2+c^3+c^4/3-0.75c^5}{2c^5}\Bigg) \\
&+18\Bigg(\frac{e^{-2c}-1+2c-2c^2}{2c^3} + \frac{20e^{-2c}+11+12c-32c^2+32c^3+e^{-4c}-32e^{-c}}{16c^4} \\
&\quad + \frac{-12e^{-c}+7-4c+4c^3/3-2c^4+8e^{-2c}+e^{-4c}-4e^{-3c}}{4c^5} \\
&\quad + \frac{e^{-4c}-24e^{-c}+22e^{-2c}-8e^{-3c}-(-9+4c^2-(13/6)c^4+2c^5)}{4c^6}\Bigg)\Bigg].
\end{aligned}$$

(3) When $0 < \alpha < 1$*,*

$$\begin{aligned}
tr(G_T) &= \frac{T^{\alpha}}{2c} + o(T^{\alpha}),\\
tr(M_T^2) &= \frac{T^{1+\alpha}}{4c} + o(T^{1+\alpha}),\\
\frac{B_{N,T}(\theta_T,\sigma^2)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} &= \frac{-3/(T+1)}{\sqrt{V(\theta_T,\sigma^2)}} + \sqrt{\frac{NT^{\alpha-1}}{2c}} + o(\sqrt{NT^{\alpha-1}}).
\end{aligned}$$

(4) when $\alpha = 0$*,*

$$\begin{aligned}
tr(G_T) &= -\frac{1}{c} + \frac{3}{1-(1-c)^2} + o(1),\\
tr(M_T^2) &= \frac{T}{2-2(1-c)^2} + o(T),\\
\frac{B_{N,T}(\theta_T,\sigma^2)}{\sqrt{V_{N,T}(\theta_T,\sigma^2)}} &= \frac{-3/(T+1)}{\sqrt{V(\theta_T,\sigma^2)}} + \frac{-1/c+3/(1-(1-c)^2)}{\sqrt{1/(1-(1-c)^2)}}\sqrt{NT^{-1}} + o(\sqrt{NT^{-1}}).
\end{aligned}$$

Proof 3. *By the Corollary 3 and 4.*

Appendix C. Proof of Theorem 1

Corollary 6 (Lemma 11 in Hahn(2002)[3] $\frac{1}{NT^2}\sum_{i=1}^{N}\varepsilon_i' D_T' D_T \varepsilon_i$**).**

(1) When $\alpha > 1$*,*

$$\frac{1}{NT^2}\sum_{i=1}^{N}\varepsilon_i' D_T' D_T \varepsilon_i = \frac{\sigma^2}{6} + o_p(1),$$

(2) When $\alpha = 1$*,*

$$\frac{1}{NT^2}\sum_{i=1}^{N}\varepsilon_i' D_T' D_T \varepsilon_i = \sigma^2 K_c + o_p(1),$$

where $K_c = \left(-\frac{1-e^{-2c}-2c+2c^2}{2c^3} + \frac{e^{-2c}-1+2c}{4c^2} - \frac{2e^{-c}-2+2c-c^2}{c^3}\right)$.

(3) When $0 < \alpha < 1$*,*

$$\begin{aligned}
\frac{1}{NT^2}\sum_{i=1}^{N}\varepsilon_i' D_T' D_T \varepsilon_i &= 0 + o_p(1),\\
\frac{1}{NT^{1+\alpha}}\sum_{i=1}^{N}\varepsilon_i' D_T' D_T \varepsilon_i &= \frac{\sigma^2}{2c} + o_p(1)
\end{aligned}$$

(4) When $\alpha = 0$,

$$\frac{1}{NT^2}\sum_{i=1}^{N}\varepsilon_i' D_T' D_T \varepsilon_i = 0 + o_p(1),$$
$$\frac{1}{NT}\sum_{i=1}^{N}\varepsilon_i' D_T' D_T \varepsilon_i = \frac{\sigma^2}{1-(1-c)^2} + o_p(1)$$

Proof: Markov Inequality and Corollary 3.

Corollary 7 (Variance Correction $V_{N,T}(\theta_T, \sigma^2)$).

(1) when $\alpha > 1$,

$$V_{N,T}(\theta_T, \sigma^2) = \frac{\frac{1}{NT^4} 2\sigma^4 tr(M_T^2)}{\left(\frac{1}{NT^2}\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})^2\right)^2} = \frac{1}{NT^2}\frac{51}{5}(1 + o_p(1))$$

(2) when $\alpha = 1$,

$$V_{N,T}(\theta_T, \sigma^2) = \frac{\frac{1}{NT^4} 2\sigma^4 tr(M_T^2)}{\left(\frac{1}{NT^2}\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})^2\right)^2} = \frac{1}{NT^2}\frac{2B_c}{K_c^2}(1 + o_p(1))$$

where B_c is from Corollary 5, K_c is defined in Corollary 6.

(3) when $0 < \alpha < 1$,

$$V_{N,T}(\theta_T, \sigma^2) = \frac{\frac{1}{NT^{2+2\alpha}} 2\sigma^4 tr(M_T^2)}{\left(\frac{1}{NT^{1+\alpha}}\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})^2\right)^2} = \frac{2c}{NT^{1+\alpha}}(1 + o_p(1))$$

(4) when $\alpha = 0$,

$$V_{N,T}(\theta_T, \sigma^2) = \frac{\frac{1}{NT^2} 2\sigma^4 tr(M_T^2)}{\left(\frac{1}{NT}\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})^2\right)^2} = \frac{1-(1-c)^2}{NT}(1 + o_p(1))$$

Proof 4. *For the moments of products of quadratic forms in normal variables, we have the matrix's trace expression in Lemma 2.3 from Magnus(1978)[12] that,*

$$\begin{aligned} &Var\left(\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})\,\varepsilon_{it} + \frac{3}{T+1}\sum_{i=1}^{N}\sum_{t=1}^{T}(y_{i,t-1} - \bar{y}_{-1,i})^2\right) \\ &= \sum_{i=1}^{N} Var(\varepsilon_i' M_T \varepsilon_i) = \sum_{i=1}^{N} 2\sigma^4 tr(M_T) = N2\sigma^4 tr(M_T) \end{aligned} \tag{C.1}$$

The rest of the proof can be easily derived from the result in Corollary 6 and Corollary 5.

Corollary 8 (lyapunov condition).

(1) When $\alpha > 1$,

$$E(\epsilon' G_T \epsilon) = \sigma^2 tr(G_T) = o(T),$$
$$E[(\epsilon' G_T \epsilon)^4] = O(T^4),$$
$$Var(\epsilon' G_T \epsilon) = Var(\epsilon' M_T \epsilon) = 2\sigma^4 tr(M_T^2) = \frac{17\sigma^4}{60} T^2 + o(T^2)$$

(2) When $\alpha = 1$,

$$E(\epsilon' G_T \epsilon) = \sigma^2 tr(G_T) = A_c \sigma^2 T + o(T),$$
$$E[(\epsilon' G_T \epsilon)^4] = O(T^4),$$
$$Var(\epsilon' G_T \epsilon) = Var(\epsilon' M_T \epsilon) = 2\sigma^4 tr(M_T^2) = 2B_c \sigma^4 T^2 + o(T^2)$$

(3) When $0 < \alpha < 1$,

$$E(\epsilon' G_T \epsilon) = \sigma^2 tr(G_T) = \frac{\sigma^2}{2c} T^\alpha + o(T),$$
$$E[(\epsilon' G_T \epsilon)^4] = O(T^{2+2\alpha}),$$
$$Var(\epsilon' G_T \epsilon) = Var(\epsilon' M_T \epsilon) = 2\sigma^4 tr(M_T^2) = \frac{\sigma^4}{2c} T^{1+\alpha} + o(T^2)$$

(4) When $\alpha = 0$,

$$E(\epsilon' G_T \epsilon) = \sigma^2 tr(G_T) = O(1),$$
$$E[(\epsilon' G_T \epsilon)^4] = O(T),$$
$$Var(\epsilon' G_T \epsilon) = Var(\epsilon' M_T \epsilon) = 2\sigma^4 tr(M_T^2) = O(T)$$

Proof: by Corollary 5.

Lemma 3. *As* $N, T \to \infty$,*we have that*

$$\frac{\hat{\theta} - \theta_T - B_{N,T}}{\sqrt{V_{N,T}}} = \frac{\sum_{i=1}^N \{\epsilon_i' G_T \epsilon_i - E[\epsilon_i' G_T \epsilon_i]\}}{\sqrt{N Var(\epsilon_i' G_T \epsilon_i)}} \xrightarrow{d} N(0,1),$$

for all $\alpha \in [0, \infty)$, $\kappa \in (0, \infty)$.

Proof 5.

$$\begin{aligned}
\frac{\hat{\theta} - \theta_T - B_{N,T}}{\sqrt{V_{N,T}}} &= \sqrt{\frac{(\sum_i \epsilon_i' D_T' D_T \epsilon_i)^2}{N 2\sigma^4 tr(M_T^2)}} \frac{\sum_i \epsilon_i' D_T \epsilon_i + \frac{3}{T+1} \sum_i \epsilon_i' D_T' D_T \epsilon_i - \sigma^2 N tr(G_T)}{\sum_i \epsilon_i' D_T' D_T \epsilon_i} \\
&= \sqrt{\frac{(\sum_i \epsilon_i' D_T' D_T \epsilon_i)^2}{Var(\sum_i \epsilon_i' G_T \epsilon_i)}} \frac{\sum_i \epsilon_i' G_T \epsilon_i - E(\sum_i \epsilon_i' G_T \epsilon_i)}{\sum_i \epsilon_i' D_T' D_T \epsilon_i} \\
&= \frac{\sum_i \epsilon_i' G_T \epsilon_i - E(\sum_i \epsilon_i' G_T \epsilon_i)}{\sqrt{Var(\sum_i \epsilon_i' G_T \epsilon_i)}}
\end{aligned}$$

We consider all possible cases of parameter regions.

*Consider Case 0. When $c = 0$, the result directly follows by **Corollary** 5.(0). Because the fourth central moment $E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^4]$ is of order T^4, and the variance is of order T^2, by Lyapunov condition*

$$\frac{\sum_{i=1}^{N} E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^4]}{Var(\sum_{i=1}^{N} \epsilon_i' G_T \epsilon_i)^2} = O\left(\frac{NT^4}{N^2 T^4}\right)$$

, we have asymptotic standard normality[3],[12].

Consider Case I, where $\alpha > 1$, $q(T) = T^\alpha / c$.

*Follow the result in **Corollary** 5.(1), since the fourth central moment $E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^4] \leq 8E[(\epsilon_i G_T \epsilon_i)^4] + 8[E(\epsilon_i G_T \epsilon_i)]^4 = O(T^4)$, and the variance $E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^2]$ is of order T^2, by Lyapunov condition, we have asymptotic standard normality.*

Consider Case II, where $\alpha = 1$, $q(T) = T/c$.

*Follow the result in **Corollary** 5.(2), since the fourth central moment $E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^4] \leq 8E[(\epsilon_i G_T \epsilon_i)^4] + 8[E(\epsilon_i G_T \epsilon_i)]^4 = O(T^4)$, and the variance $E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^2]$ is of order T^2, by Lyapunov condition, we have asymptotic standard normality.*

Consider Case III, where $0 < \alpha < 1$.

*Follow the result in **Corollary** 5.(3), since the fourth central moment $E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^4] \leq 8E[(\epsilon_i G_T \epsilon_i)^4] + 8[E(\epsilon_i G_T \epsilon_i)]^4 = O(T^{2+2\alpha})$, and the variance $E[(\epsilon_i G_T \epsilon_i - E\epsilon_i G_T \epsilon_i)^2]$ is of order $T^{1+\alpha}$, by Lyapunov condition, we have asymptotic standard normality.*

Similarly in the case IV, where $\alpha = 0$.

Proof 6 (The proof of uniform convergence). *Given the path-wise convergence from Lemma 3, we could simply follow the proof of Theorem 3.2 in Chao and Philips (2019) [6] using Lehmann (2004) Lemma 2.6.2.*

Appendix D. Proof of Theorem 2: $\boldsymbol{\sigma^2}$ estimation

Proof 7. *Let $\hat{e}_{it} := y_{it} - \hat{\theta} y_{it-1} - \hat{\alpha}_i$ denote the residuals. Since $\hat{\alpha}_i = \sum_t (y_{it} - \hat{\theta} y_{it-1})/T$, we have*

$$\hat{e}_{it} = y_{it} - \bar{y}_i - \hat{\theta}(y_{it-1} - \bar{y}_{-i});$$

Because $y_{it} - \bar{y}_i = \theta_T (y_{it-1} - \bar{y}_{-i}) + (\varepsilon_{it} - \bar{\varepsilon}_i)$ and $\hat{e}_{it} - (\varepsilon_{it} - \bar{\varepsilon}_i) = (\hat{\theta} - \theta_T)(y_{it-1} - \bar{y}_{-i})$,

$$\begin{aligned}\hat{\sigma}^2 &= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{e}_{it}^2 = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} [(\varepsilon_{it} - \bar{\varepsilon}_i) + (\hat{\theta} - \theta_T)(y_{it-1} - \bar{y}_{-i})]^2 \\ &= \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\varepsilon_{it} - \bar{\varepsilon}_i)^2 + \frac{(\hat{\theta} - \theta_T)^2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it-1} - \bar{y}_{-i})^2 + \frac{2(\hat{\theta} - \theta_T)}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (\varepsilon_{it} - \bar{\varepsilon}_i)(y_{it-1} - \bar{y}_{-i})\end{aligned}$$

So that,

$$\frac{\hat{\sigma}^2}{\sigma^2} - 1 = \left\{ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\frac{\varepsilon_{it} - \bar{\varepsilon}_i}{\sigma} \right)^2 - 1 \right\} + \left\{ \frac{(\hat{\theta} - \theta_T)^2}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\frac{y_{it-1} - \bar{y}_{-i}}{\sigma} \right)^2 \right\} +$$

$$\left\{ \frac{2(\hat{\theta} - \theta_T)}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\frac{\varepsilon_{it} - \bar{\varepsilon}_i}{\sigma} \right) \left(\frac{y_{it-1} - \bar{y}_{-i}}{\sigma} \right) \right\}$$

$$= \left\{ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\frac{\varepsilon_{it} - \bar{\varepsilon}_i}{\sigma} \right)^2 - 1 \right\} + \left\{ \frac{\frac{(\sum_{i=1}^{N} \varepsilon_i' D_T \varepsilon_i)^2}{(\sum_{i=1}^{N} \varepsilon_i' D_T' D_T \varepsilon_i)^2}}{NT} \frac{\sum_{i=1}^{N} \varepsilon_i' D_T' D_T \varepsilon_i}{\sigma^2} \right\} +$$

$$\left\{ \frac{2 \frac{\sum_{i=1}^{N} \varepsilon_i' D_T \varepsilon_i}{\sum_{i=1}^{N} \varepsilon_i' D_T' D_T \varepsilon_i}}{NT} \frac{\sum_{i=1}^{N} \varepsilon_i' D_T \varepsilon_i}{\sigma^2} \right\}$$

$$= \left\{ \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(\frac{\varepsilon_{it} - \bar{\varepsilon}_i}{\sigma} \right)^2 - 1 \right\} + \frac{3}{NT\sigma^2} \left\{ \frac{(\sum_{i=1}^{N} \varepsilon_i' D_T \varepsilon_i)^2}{\sum_{i=1}^{N} \varepsilon_i' D_T' D_T \varepsilon_i} \right\}$$

By Corollary 6, $\sum_{i=1}^{N} \varepsilon_i' D_T' D_T \varepsilon_i = O_p(NT^2)$*. And by Lemma 3,* $\sum_{i=1}^{N} \varepsilon' D_T \varepsilon = O_p(\sqrt{NT^2})$*. So that it is easy to see that the two terms converge to zero in probability uniformly over* Θ_T *and uniformly over all values of* $\sigma > 0$*.*

Appendix E. Proof of Theorem 3

Proof 8. *For* $V_{N,T}(\theta_T, \sigma^2) = \frac{2N\sigma^4 tr[M_T^2(\theta_T)]}{\left(\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}(y_{i,t-1}-\bar{y}_{-1,i})^2\right)^2}$, $\hat{V}_{N,T}(\theta_T) = \frac{2N\hat{\sigma}^4 tr[M_T^2(\theta_T)]}{\left(\sum\limits_{i=1}^{N}\sum\limits_{t=1}^{T}(y_{i,t-1}-\bar{y}_{-1,i})^2\right)^2}$

$$\frac{V_{N,T}(\theta_T, \sigma^2)}{\hat{V}_{N,T}(\theta_T)} = \frac{2N\sigma^4 tr[M_T^2(\theta_T)]}{2N\hat{\sigma}^4 tr[M_T^2(\theta_T)]} = \frac{\sigma^4}{\hat{\sigma}^4} \to 1.$$

And so by Lemma 3 and Continuous mapping theorem, we have the path-wise convergence,

$$\hat{t}_{N,T}(\theta_T) = \frac{\hat{\theta} - \theta_T - \hat{B}_{N,T}(\theta_T)}{\sqrt{\hat{V}_{N,T}(\theta_T)}} \xrightarrow{d} N(0,1). \tag{E.1}$$

Then we follow the uniform convergence proof as in proof 6.